The purpose of this course is to introduce the basic physics and essential tools in theoretical cosmology. The course starts by introducing the relativistic perturbation theory and focuses on its applications to large-scale structure probes of inflationary cosmology such as the cosmic microwave background radiation, galaxy clustering, and weak gravitational lensing. We will also study the origin of the perturbation generation in the early Universe. It is recommended to have good understanding of general relativity and quantum field theory.
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1 Newtonian Perturbation Theory

1.1 Standard Newtonian Perturbation Theory

1.1.1 Summary of the Governing Equations

In Newtonian dynamics, fully nonlinear equation for a pressureless fluid (CDM and baryons) can be written down:

\[ \dot{\delta} + \frac{1}{a} \nabla \cdot \mathbf{v} = -\frac{1}{a} \nabla \cdot (\mathbf{v} \delta), \quad \nabla \cdot \mathbf{v} = H \nabla \cdot \mathbf{v} + \frac{3H^2}{2a} \dot{\Omega}_m \delta = -\frac{1}{a} \nabla \cdot ([\mathbf{v} \cdot \nabla] \mathbf{v}), \quad \nabla^2 \phi = 4\pi G \rho. \]  

(1.1)

The Euler equation can be split into one for divergence and one for vorticity. The vorticity vector \( \nabla \times \mathbf{v} \) decays at the linear order. At nonlinear level, if no anisotropic pressure and no initial vorticity, the vorticity vanishes at all orders. However, in reality, the anisotropic pressure arises from shell crossing, generating vorticity on small scales, even in the absence of the initial vorticity. Of course, baryons are not exactly pressureless; they form galaxies, and their feedback effects are also important up to fairly large scales. These all modify the SPT equation.

- regime of validity, measurement precision, analytic vs numerical simulations, galaxy surveys

1.1.2 Basic Formalism

We consider multi-component fluids in the presence of isotropic pressure. In case of \( n \)-fluids with the mass densities \( \rho_i \), the pressures \( p_i \), the velocities \( \mathbf{v}_i \) \((i = 1, 2, \ldots n)\), and the gravitational potential \( \Phi \), we have

\[ \dot{\rho}_i + \nabla \cdot (\rho_i \mathbf{v}_i) = 0, \quad \dot{\mathbf{v}}_i + \mathbf{v}_i \cdot \nabla \mathbf{v}_i = -\frac{1}{\rho_i} \nabla p_i - \nabla \Phi, \quad \nabla^2 \Phi = 4\pi G \sum_{j=1}^{n} \rho_j. \]

(1.2)

Assuming the presence of spatially homogeneous and isotropic but temporally dynamic background, we introduce fully nonlinear perturbations as

\[ \dot{\rho}_i = \bar{\rho}_i + \delta \rho_i, \quad p_i = \bar{p}_i + \delta p_i, \quad \mathbf{v}_i = H \mathbf{r} + \mathbf{u}_i, \quad \Phi = \bar{\Phi} + \delta \Phi, \]

(1.3)

where \( H := \dot{a}/a \), and \( a(t) \) is a cosmic scale factor. We move to the comoving coordinate \( \mathbf{x} \) where

\[ r := a(t) \mathbf{x}, \quad \nabla = \nabla_r = \frac{1}{a} \nabla_x, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial t} \bigg|_r = \frac{\partial}{\partial t} \bigg|_r + \left( \frac{\partial}{\partial t} \bigg|_r \mathbf{x} \right) \cdot \nabla_x = \frac{\partial}{\partial t} \bigg|_x - H \mathbf{x} \cdot \nabla_x. \]

(1.4)

In the following we neglect the subindex \( x \). To the background order we have

\[ \dot{\bar{\rho}}_i + 3H \bar{\rho}_i = 0, \quad H^2 + \dot{H} = \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \sum_j \theta_j, \quad H^2 = \frac{8\pi G}{3} \sum_j \theta_j + \frac{2E}{a^2}, \]

(1.5)

where \( E \) is an integration constant which can be interpreted as the specific total energy in Newton’s gravity; in Einstein’s gravity we have \( 2E = -Kc^2 \) where \( K \) can be normalized to be the sign of spatial curvature. Note the difference in the background equation in Newtonian cosmology. The nonlinear governing equations can be expressed in terms of the perturbation variables as

\[ \dot{\delta}_i + \frac{1}{a} \nabla \cdot \mathbf{u}_i = -\frac{1}{a} \nabla \cdot (\delta_i \mathbf{u}_i), \quad \frac{1}{a^2} \nabla^2 \dot{\Phi} = 4\pi G \sum_j \theta_j \delta_j, \]

(1.6)

\[ \mathbf{u}_i + H \mathbf{u}_i + \frac{1}{a} \mathbf{u}_i \cdot \nabla \mathbf{u}_i = -\frac{1}{a \bar{\rho}_i} \nabla p_i - \frac{1}{a} \nabla \delta \Phi. \]

(1.7)

By introducing the expansion \( \theta_i \) and the rotation \( \vec{\omega}_i \) of each component as

\[ \theta_i := -\frac{1}{a} \nabla \cdot \mathbf{u}_i, \quad \vec{\omega}_i := \frac{1}{a} \nabla \times \mathbf{u}_i, \]

(1.8)
we derive

\[
\dot{\delta}_i + 2H\theta_i - 4\pi G \sum_j \phi_j \delta_j = \frac{1}{a^2} \nabla \cdot (\mathbf{u}_i \cdot \nabla \mathbf{u}_i) + \frac{1}{a^2 \theta_i} \nabla \cdot \left( \frac{\nabla \delta p_i}{1 + \delta_i} \right),
\]

(1.9)

\[
\dot{\omega}_i + 2H \omega_i = -\frac{1}{a^2} \nabla \times (\mathbf{u}_i \cdot \nabla \mathbf{u}_i) + \frac{1}{a^2 \theta_i} \left( \nabla \delta_i \times \nabla \delta p_i \right).
\]

(1.10)

By introducing decomposition of perturbed velocity into the potential- and transverse parts as

\[
\mathbf{u}_i := -\nabla U_i + \mathbf{u}^{(v)}_i, \quad \nabla \cdot \mathbf{u}^{(v)}_i \equiv 0, \quad \theta_i = \frac{\Delta}{a} U_i, \quad \omega_i = \frac{1}{a} \nabla \times \mathbf{u}^{(v)}_i,
\]

(1.11)

we have

\[
\ddot{\mathbf{u}}^{(v)}_i + H \dot{\mathbf{u}}^{(v)}_i = -\frac{1}{a} \left[ \mathbf{u}_i \cdot \nabla \mathbf{u}_i + \frac{1}{a} \nabla \delta p_k \right] - \nabla \Delta^{-1} \nabla \cdot \left( \mathbf{u}_i \cdot \nabla \mathbf{u}_i + \frac{1}{a} \nabla \delta p_i \right).
\]

(1.12)

Combining equations above, we can derive

\[
\ddot{\delta}_i + 2H \dot{\delta}_i - 4\pi G \sum_j \phi_j \delta_j = -\frac{1}{a^2} \left[ a \nabla \cdot (\delta \mathbf{v}) \right] + \frac{1}{a^2} \nabla \cdot (\mathbf{u}_i \cdot \nabla \mathbf{u}_i) + \frac{1}{a^2 \theta_i} \nabla \cdot \left( \frac{\nabla \delta p_i}{1 + \delta_i} \right).
\]

(1.13)

These equations are valid to fully nonlinear order. The density fluctuation grows against the Hubble friction. Notice that for vanishing pressure these equations have only quadratic order nonlinearity in perturbations.

• numerical simulations, baryons

1.1.3 Linear-Order and Second-Order Solutions

We will derive the solutions for a single pressureless medium (now we change notation \( \mathbf{u}_i \rightarrow \mathbf{v} \))

\[
\ddot{\delta} + \frac{1}{a} \nabla \cdot \mathbf{v} = -\frac{1}{a} \nabla \cdot (\delta \mathbf{v}), \quad \dot{\theta} + 2H\theta - 4\pi G \ddot{\delta} = \frac{1}{a^2} \nabla \cdot (\mathbf{v} \cdot \nabla \mathbf{v}),
\]

(1.14)

where we now use \( \mathbf{v} \) to represent the velocity perturbation. The calculations are greatly simplified in Fourier space, and our convention is

\[
A(x) \equiv \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k} \cdot \mathbf{x}} A(k), \quad A(k) \equiv \int d^3x e^{-i\mathbf{k} \cdot \mathbf{x}} A(x),
\]

(1.15)

and we often use the identity:

\[
\delta^D(k) = \int \frac{d^3x}{(2\pi)^3} e^{i\mathbf{k} \cdot \mathbf{x}}.
\]

(1.16)

First, we derive the linear-order solution. At the linear order in perturbations, we can separate the time-dependence and the spatial-dependence, i.e., all different Fourier modes evolve at the same rate, and the growth rate \( D \) satisfies

\[
\ddot{D} + 2H \dot{D} - 4\pi G \ddot{\delta} = D(t) \delta(k), \quad \dot{D}(t) = \frac{D_1(t)}{D_1(t_0)}.
\]

(1.17)

where the (dimensionless) growth factor \( D(t) \) is normalized to unity at some early epoch \( t_0 \) when the nonlinearities are ignored \( \delta(t_0, \mathbf{k}) := \delta^{(1)}(t_0, \mathbf{k}) \equiv \delta(k) \). The linear-order solution for the density and the velocity divergence is then

\[
\delta^{(1)}(t, \mathbf{k}) = D(t) \delta(k), \quad \theta^{(1)}(t, \mathbf{k}) = H f D(t) \delta(k), \quad f := \frac{d \ln D}{d \ln a},
\]

(1.18)

where the superscript indicates the perturbation order, the logarithmic growth rate \( f \) is approximately time-independent and it is unity \( f = 1 \) in the matter-dominated era.
To derive the second-order solution, we need to Fourier decompose the source terms in the right-hand side of the dynamical equation. At the second-order in perturbations, the quadratic terms represent the product of two linear-order terms. Furthermore, the quadratic product in configuration space becomes the convolution in Fourier space:

\[
\left\{ \frac{1}{a^2} \nabla \cdot [(v \cdot \nabla)v] \right\} \quad \Rightarrow \quad \mathcal{H} f \frac{D^2}{D^2} \int \frac{d^3 q_1}{(2\pi)^3} \int \frac{d^3 q_2}{(2\pi)^3} (2\pi)^3 \delta^D(k - Q_{12}) \left(1 + \frac{Q_1 \cdot Q_2}{Q_1^2} \right) \delta(Q_1) \delta(Q_2), \quad (1.19)
\]

where \(Q_{12} = Q_1 + Q_2\). Using the source functions in Fourier space, we can solve the governing equations for the density and the velocity divergence as

\[
\frac{\delta^{(2)}(t, k)}{D^2} = \int \frac{d^3 q_1}{(2\pi)^3} \int \frac{d^3 q_2}{(2\pi)^3} (2\pi)^3 \delta^D(k - q_{12}) \delta(q_1) \delta(q_2) \left[ \frac{5}{7} + \frac{2}{7} \left( \frac{q_1 \cdot q_2}{q_1^2 q_2^2} \right) + \frac{q_1 \cdot q_2}{2 q_1 q_2} \left( \frac{q_1}{q_2} + \frac{q_2}{q_1} \right) \right], \quad (1.21)
\]

\[
\frac{\theta^{(2)}(t, k)}{H f D^2} = \int \frac{d^3 q_1}{(2\pi)^3} \int \frac{d^3 q_2}{(2\pi)^3} (2\pi)^3 \delta^D(k - q_{12}) \delta(q_1) \delta(q_2) \left[ \frac{3}{7} + \frac{4}{7} \left( \frac{q_1 \cdot q_2}{q_1^2 q_2^2} \right) + \frac{q_1 \cdot q_2}{2 q_1 q_2} \left( \frac{q_1}{q_2} + \frac{q_2}{q_1} \right) \right]. \quad (1.22)
\]

- **HW:** Derive the second-order solutions

### 1.1.4 General Solution

Beyond the linear order, the density and the velocity divergence grow in a nonlinear fashion, i.e., different Fourier modes couple. By assuming the separability of the time and the spatial dependences, the standard perturbation theory (SPT) takes a perturbative approach to the nonlinear solution:

\[
\delta(t, k) := \sum_{n=1}^{\infty} D^n(t) \left[ \prod_i \int \frac{d^3 q_i}{(2\pi)^3} \delta(q_i) \right] (2\pi)^3 \delta^D(k - q_{12-\ldots n}) \equiv \sum_{n=1}^{\infty} D^n(t) \delta^{(n)}(k), \quad (1.23)
\]

\[
\frac{\theta(t, k)}{H f} := \frac{\sum_{n=1}^{\infty} D^n(t) \left[ \prod_i \int \frac{d^3 q_i}{(2\pi)^3} \delta(q_i) \right] (2\pi)^3 \delta^D(k - q_{12-\ldots n}) G_n^{(s)}(q_1, \ldots, q_n)}{\sum_{n=1}^{\infty} D^n(t) \theta^{(n)}(k)} \quad (1.24)
\]

where \(q_{12-\ldots n} \equiv q_1 + \ldots + q_n\), \(\delta^{(n)}(k)\) and \(\theta^{(n)}(k)\) are time-independent \(n\)-th order perturbations, \(F_n^{(s)}\) and \(G_n^{(s)}\) are the SPT kernels symmetrized over its arguments. With these decompositions in Fourier space, the LHS of the Newtonian dynamical equations become

\[
\frac{\ddot{\delta}}{H} = H f \sum_{n=1}^{\infty} D^n \left( n \delta^{(n)} - \theta^{(n)} \right), \quad \dot{\theta} = 2H \theta - 4\pi G \beta_0 \delta = H f \frac{D^2}{2} \left[ 1 + 2n \theta^{(n)} - 3\delta^{(n)} \right], \quad (1.25)
\]

where we utilized the relation between the growth factor and the growth rate \(\dot{\delta} = H D f\). The RHS of the Newtonian dynamical equations are the convolution in Fourier space:

\[
\left\{ \frac{1}{a^2} \nabla \cdot (v \cdot D) \right\} (k) = \int \frac{d^3 Q_1}{(2\pi)^3} \int \frac{d^3 Q_2}{(2\pi)^3} (2\pi)^3 \delta^D(k - Q_{12}) \alpha_{12} \delta(Q_1, t) \delta(Q_2, t) \equiv H f \sum_{n=1}^{\infty} D^n A_n(k), \quad (1.26)
\]

\[
\left\{ \frac{1}{a^2} \nabla \cdot (v \cdot \nabla) \right\} (k) = \int \frac{d^3 Q_1}{(2\pi)^3} \int \frac{d^3 Q_2}{(2\pi)^3} (2\pi)^3 \delta^D(k - Q_{12}) \beta_{12} \delta(Q_1, t) \theta(Q_2, t) \equiv H f \frac{D^2}{2} \sum_{n=1}^{\infty} D^n B_n(k), \quad (1.27)
\]

where the vertex functions are defined as

\[
\alpha_{12} := \alpha(Q_1, Q_2) \equiv 1 + \frac{Q_1 \cdot Q_2}{Q_1^2}, \quad \beta_{12} := \beta(Q_1, Q_2) \equiv \frac{|Q_1 + Q_2|^2}{2Q_1^2 Q_2^2}, \quad (1.28)
\]

and the \(n\)-th order perturbation kernels \(A_n(k)\) and \(B_n(k)\) are

\[
A_n(k) = \left[ \prod_i \int \frac{d^3 q_i}{(2\pi)^3} \delta(q_i) \right] (2\pi)^3 \delta^D(k - q_{12-\ldots n}) \sum_{i=1}^{n-1} \alpha_{12} G_i(q_1, \ldots, q_i) F_{n-i}(q_{i+1}, \ldots, q_n), \quad (1.29)
\]

\[
B_n(k) = \left[ \prod_i \int \frac{d^3 q_i}{(2\pi)^3} \delta(q_i) \right] (2\pi)^3 \delta^D(k - q_{12-\ldots n}) \sum_{i=1}^{n-1} \beta_{12} G_i(q_1, \ldots, q_i) G_{n-i}(q_{i+1}, \ldots, q_n), \quad (1.30)
\]
with $Q_1 = q_{1..i}$ and $Q_1 + Q_2 = k$.

Therefore, the two Newtonian dynamical equations become algebraic equations with the time-dependence removed:

$$n\delta^{(n)} - \theta^{(n)} = A_n,$$
$$\quad (1 + 2n)\theta^{(n)} - 3\delta^{(n)} = 2B_n,$$  \hspace{1cm} (1.31)

and the well-known recurrence formulas for the solutions are

$$\delta^{(n)} = \frac{(1 + 2n)A_n + 2B_n}{(2n + 3)(n - 1)}; \quad \theta^{(n)} = \frac{3A_n + 2nB_n}{(2n + 3)(n - 1)},$$ \hspace{1cm} (1.32)

and similarly so for the SPT kernels

$$F_n = \sum_{i=1}^{n-1} \frac{G_i}{(2n + 3)(n - 1)} [(1 + 2n)\alpha_{12} F_{n-i} + 2\beta_{12} G_{n-i}],$$ \hspace{1cm} (1.33)

$$G_n = \sum_{i=1}^{n-1} \frac{G_i}{(2n + 3)(n - 1)} [3\alpha_{12} F_{n-i} + 2n\beta_{12} G_{n-i}], \quad F_1 = G_1 = 1.$$ \hspace{1cm} (1.34)


# 2 Probes of Inhomogeneity

In cosmology, the initial condition is set in the early Universe with Gaussian random fluctuations in Fourier space, as the quantum fluctuations in vacuum are stretched beyond the horizon scales during the inflationary epoch. Since the Gaussian distribution is completely specified by the variance, the power spectrum contains all the information in the early Universe. However, the nonlinear growth in the late time complicates the interpretations. Here we focus on the linear theory and study various ways to measure the two-point statistics.

## 2.1 Basic Formalism

### 2.1.1 Two-Point Correlation Function and Power Spectrum

- **3D information.** Suppose that we use some cosmological probes such as galaxies and measure, say, the matter density fluctuation $\delta$. Now imagine we have measurements of such probe over all positions $x$. We can then measure the two-point correlation function $\xi(r)$ and its Fourier transform, the power spectrum $P(k)$:

$$
\xi(r) = \langle \delta(x)\delta(x+r) \rangle = \int \frac{d^3k}{(2\pi)^3} e^{ik\cdot r} P(k),
$$

and the variance is

$$
\sigma^2 = \xi(0) = \int d\ln k \frac{k^3}{2\pi^2} P(k), \quad \Delta^2_k := \frac{k^3}{2\pi^2} P(k),
$$

where $\Delta^2_k$ is the dimensionless power spectrum and it is the contribution to the variance per each log $k$.

Note that different Fourier modes are not correlated in the initial condition and the power spectrum characterizes the Gaussian distribution at each Fourier mode. Therefore, using cosmological probes, we need to measure the distribution map $\delta(x)$ and compute the two-point correlation function or the power spectrum.

- **1D information.** Spectroscopic measurements of distant quasars yield the density fluctuations of neutral hydrogens along the line-of-sight. In this case, we probe the density fluctuation, but only in terms of the line-of-sight separation, say, $z$-direction. Given the 1D map, we can measure the 1D correlation function, and it is related to the power spectrum as

$$
\xi_{1D}(z) = \langle \delta(x)\delta(x+z) \rangle = \int \frac{d^3k}{(2\pi)^3} e^{ik_z z} P_{3D}(k),
$$

where the separation vector is $z = z\hat{z}$ along the line-of-sight direction. We can also define 1D power spectrum that is a Fourier counterpart of the 1D correlation function:

$$
P_{1D}(k_z) \equiv \int dz \ e^{-ik_z z} \xi_{1D}(z) = \int \frac{d^3k'}{(2\pi)^3} P(k')\delta_{1D}(k_z', k_z) = \int_0^{\infty} \frac{dk'}{2\pi} k' P(k', k_z) = \int_{k_z}^{\infty} \frac{dk}{2\pi} k P(k),
$$

where we again assumed that the 3D power spectrum is isotropic. The 1D power spectrum is the projection of the 3D power spectrum over 2D Fourier space. For sufficiently high $k$, it is largely one-to-one, though it has bias (called aliasing) on low $k$. This relation can be inverted as

$$
P(k) = -\frac{2\pi}{k} \frac{d}{dk} P_{1D}(k),
$$

and the dimensionless power spectrum in 1D is

$$
\sigma^2_{1D} = \int d\ln k_z \frac{k_z}{\pi} P_{1D}(k_z), \quad \Delta^2_{k,1D} := \frac{k_z}{\pi} P_{1D}(k_z).
$$

1However, as we studied in Section 1, the nonlinear evolution results in the mode coupling.
• **2D information.** — Though the distance in cosmology is difficult to measure, it is easy to have 2D information on the sky. We define the 2D power spectrum in a similar way as the Fourier counterpart of the 2D correlation function:

$$P_{2D}(k_x, k_y) \equiv \int dx \int dy \ e^{-ik_x x} e^{-ik_y y} \xi_{2D}(x, y) = \int \frac{dk'}{2\pi} \ P(k_x, k_y, k_y') = \frac{1}{\pi} \int_{k_\perp}^{\infty} \frac{dk'}{k_\perp} \ \frac{k' P(k')}{k'^2 - k^2_\perp}, \quad (2.7)$$

where $k^2_\perp = k_x^2 + k_y^2$. The 2D power spectrum is the projection over 1D Fourier space, and its similar relation to the 3D power spectrum exists. This relation can be again inverted by using the (non-trivial) Abell integral as

$$P(k) = -\frac{2}{k} \int_{k}^{\infty} \frac{dk_\perp}{k_\perp} \ \frac{P_{2D}(k_\perp)}{k_\perp^2 - k^2}, \quad (2.8)$$

and the dimensionless power spectrum in 2D is then

$$\sigma^2_{2D} = \int d\ln k_\perp \frac{k^2}{2\pi} \ P_{2D}(k_\perp), \quad \Delta^2_{k,2D} = \frac{k^2}{2\pi} P_{2D}(k_\perp). \quad (2.9)$$

The projection-slice theorem says Fourier transformation of the projection is the slice of its Fourier transformation. It means exactly what we derived here. A similar relation holds in configuration space. The projected correlation function is related as

$$w_p(r_p) := \int dz \ \xi(r_p, z) = 2 \int_{r_p}^{\infty} dr \ \frac{\xi(r)}{\sqrt{r^2 - r_p^2}}, \quad \xi(r) = -\frac{1}{\pi} \int_{r}^{\infty} dr_p \ \frac{w_p(r_p)}{\sqrt{r_p^2 - r^2}}. \quad (2.10)$$

### 2.1.2 Angular Correlation and Angular Power Spectrum

We briefly covered the statistics in a flat space. However, the sky is round, and we can only make observations by measuring the light signals. The cosmic microwave background anisotropies, for example, are measured only as a function of the angular position on the sky at the Earth. In cosmology, we often have angular information, but no distance measurements. Since this measurement $\delta(\hat{\theta})$ is defined on a unit sphere, we can decompose it in terms of spherical harmonics as

$$\delta(\hat{\theta}) := \sum_{l,m} a_{lm} Y_{lm}(\hat{\theta}), \quad a_{lm} = \int d^2 \hat{\theta} \ Y_{lm}^*(\hat{\theta}) \delta(\hat{\theta}), \quad (2.11)$$

where we have discrete sum, instead of integral in Fourier space. The reality condition for $\delta$ imposes

$$a^*_{lm} = (-1)^m a_{l,-m}. \quad (2.12)$$

Similar to the case in 3D, we can define the angular correlation function and its Fourier counterpart:

$$w(\hat{\gamma}) = \langle \delta(\hat{\theta}) \delta(\hat{\theta} + \hat{\gamma}) \rangle = \sum_{lm} C_l Y_{lm}(\hat{\theta}) Y_{lm}^*(\hat{\theta} + \hat{\gamma}) = \sum_{l} \frac{2l + 1}{4\pi} C_l L_l(\cos \gamma), \quad (2.13)$$

where we used the relation

$$\langle a_{lm} a^*_{l'm'} \rangle = \delta_{ll'} \delta_{mm'} C_l = \sum_{m} \frac{|a_{lm}|^2}{2l + 1} \delta_{ll'} \delta_{mm'}, \quad (2.14)$$

and the Legendre polynomial is related to the spherical harmonics as

$$L_l(\mu) = \sum_{m=-l}^{l} \frac{4\pi}{2l + 1} Y_{lm}(\hat{\theta}_1) Y_{lm}^*(\hat{\theta}_2), \quad \mu = \hat{\theta}_1 \cdot \hat{\theta}_2. \quad (2.15)$$

The angular power spectrum can be obtained as

$$C_l = 2\pi \int_{-1}^{1} d\mu \ L_l(\mu) w(\theta). \quad (2.16)$$
2.1.3 Flat-Sky Approximation

When the area of interest is relatively small in the sky, we can use the flat-sky approximation, and it often overlaps with the distant-observer approximation, in which the observer is so far away that the position angle is virtually constant, compared to their relative positions. In this case, the angular correlation and its power spectrum are closely related to those in flat space.

Now consider the 2D correlation function $\xi_{2D}$ and 2D power spectrum $P_{2D}(k)$:

$$
\xi_{2D}(x, y) = \int \frac{d^2k_\perp}{(2\pi)^2} e^{ik_\perp \cdot x_\perp} P_{2D}(k_\perp) = \int \frac{d^2l}{(2\pi)^2} e^{il \cdot \theta} P_l, \quad P_l \equiv \frac{1}{r^2} P_{2D} \left( k_\perp = \frac{l}{r} \right),
$$

(2.17)

where we used $x_\perp = r \theta$ and defined the (flat-sky) angular power spectrum $P_l$. Note that the 2D power spectrum is dimensionful, but the angular power spectrum is dimensionless. Given the radial distance $r$, the 2D correlation function $\xi_{2D}$ can be considered as the angular correlation function, and assuming that the angular power spectrum is independent of its direction, we can further simply the relation:

$$
w(\theta) = \xi_{2D}(r \theta) = \int \frac{dl}{2\pi} l P_l J_0(l \theta),
$$

(2.18)

where $J_0$ is the Bessel function. The (full-sky) angular power spectrum is then obtained as

$$
C_l = 2\pi \int d\mu \, L_\perp(\mu) w(\theta) \approx \sum_l l P_l \frac{2 \delta w}{2l + 1} \approx P_l,
$$

(2.19)

where we manipulated the Bessel function for $l \gg 1$ and $\theta \ll 1

$$
J_0(l \theta) = \frac{1}{\pi} \int_0^\pi d\phi \, e^{i l \theta \cos \phi} \approx \frac{1}{\pi} \int_0^\pi d\phi \left( 1 + \frac{i l \theta \cos \phi}{l} \right)^l \approx \frac{1}{\pi} \int_0^\pi d\phi \, (\cos \theta + i \sin \theta \cos \phi)^l = L_l(\cos \theta).
$$

(2.20)

The angular quantities such as $w(\theta)$ and $C_l$ are defined on a unit sphere, whereas the 2D quantities such as $\xi_{2D}$ and $P_{2D}$ are defined on a 2D flat space. Hence, the former is related to each other via spherical harmonics, and the latter via Fourier transformation. But they are defined in a way that the angular power spectrum $C_l$ and its flat-sky counterpart $P_l$ are equivalent in the limit of small sky.

2.1.4 Projection and Limber Approximation

We often measure some angular quantities in cosmology, but they are often the projection of the 3D quantities. For example, one can measure the angular map in a given galaxy survey, but the angular quantity $\delta_2(\theta)$ we measure indeed derives from the 3D quantity $\delta(x)$, but projected along the line-of-sight direction with some weighting $W(r)$:

$$
\delta_2(\theta) = \int dr \, W(r) \delta(x), \quad x = (r \theta, r).
$$

(2.21)

The weight function is normalized to unity and it is often parametrized in terms of redshift as

$$
1 = \int dr \, W_r(r) = \int dz \, W_z(z),
$$

(2.22)

where the weight function can be dimensionful, depending on its parametrization. The angular correlation is then

$$
w(\theta) = \langle \delta_2(0) \delta_2(\theta) \rangle = \int dr_1 \, W(r_1) \int dr_2 \, W(r_2) \xi_{3D}(r), \quad r \approx (r_1 \theta, r_2 - r_1),
$$

(2.23)

where we assumed the flat-sky approximation. The angular power spectrum is

$$
P_l = \int \frac{d^2\theta}{(2\pi)^2} e^{-i \theta} w(\theta) = \int dr_1 \, W(r_1) \int dr_2 \, W(r_2) \int \frac{dk}{2\pi} e^{-i k \cdot (r_2 - r_1)} \frac{1}{r_1^2} P \left[ \left( k_\perp = \frac{l}{r_1}, k_z \right) \right] .
$$

(2.24)
Since we work in the flat-sky regime (or the distant observer), the radial distance is far larger than the transverse separation \( r \gg r\theta \). Hence, we have the separation of scale in Fourier space

\[ k_\perp = \frac{l}{r} \approx \frac{1}{r\theta} \gg \frac{1}{r} \approx k_z , \]  

and the power spectrum can be approximated as

\[ P(k) \approx P(k = k_\perp) + \frac{dP}{dk_z} k_z + O(k_z)^2 , \quad \frac{dP}{dk_z} k_z = \frac{dP}{dk} k^2 \approx \frac{P}{k^2} \ll P . \]  

Keeping the leading term in the power spectrum, we can integrate over \( k_z \) and approximate the angular power spectrum as

\[ P_l \approx \int dr W^2(r) \frac{r^2}{\pi} P \left( k_\perp = \frac{l}{r_1}, k_z \right) . \]  

This is sometimes called the Limber approximation. When the window function is sufficiently broad compared to the coherent length scale of the correlation, the Limber approximation is very accurate and useful. Its relation to the angular correlation is

\[ w(\theta) = \int \frac{dl}{2\pi} l P_l(l\theta) \equiv \int dk k P(k) F(k, \theta) , \]  

where we defined the kernel

\[ F(k, \theta) := \int \frac{dr}{2\pi} W^2(r) J_0(kr\theta) = \frac{1}{k} \int \frac{dl}{2\pi} W^2 \left( \frac{l}{k} \right) J_0(kr\theta) . \]  

### 2.2 Peculiar Velocity

#### 2.2.1 Observations of Peculiar Velocities

The distant objects such as galaxies are receding from us due to the Hubble expansion, and this expansion (or the receding velocity \( v \)) is measured by the redshift \( z \) of the known line-emissions from the distant objects:

\[ v \equiv cz . \]  

However, in addition to the Hubble expansion \( v_H \), these objects are also moving, and this motion is referred to as the peculiar motion \( v_p \). Due to the peculiar motion, the Doppler effect also contributes to the receding velocity, and the receding velocity can be written as

\[ v = v_H + v_p , \quad v_H = H d = \mathcal{H} r , \]  

where the object is assumed to be at the physical distance \( d \) (or comoving distance \( r \)). The redshift measurements (or the receding velocity) yield only the radial component of the receding velocity. The tangential peculiar motion can be measured. However, since this requires measurements of the angular motion of the distant objects over a long time, it is practically limited to the nearby objects such as stars in our own Galaxy. The measurements of the radial peculiar velocity also requires precise measurements of the distance \( d \), which is very difficult in cosmology. For example, 10% error in the distance measurements at \( d = 50 h^{-1} \text{Mpc} \) yields the error of 500 km s\(^{-1}\) in the peculiar velocity measurement. Therefore, the peculiar velocity measurements are also limited to the low-redshift objects.

- receding velocity at \( z > 1 \), gauge ambiguity, SN Ia or SZ measurements
- HW: derive Eq. (2.30) from Eq. (2.31)

#### 2.2.2 Linear Theory

In Chapter 1, we learned that the velocity divergence is related to the density fluctuation:

\[ \theta \equiv -\frac{1}{a} \nabla \cdot v = H f \delta . \]  

\[ \theta \equiv -\frac{1}{a} \nabla \cdot v = H f \delta . \]
Ignoring the vector perturbation, the velocity can be expressed in terms of the velocity potential $U$ as

$$\mathbf{v} = -\nabla U, \quad \theta = \frac{1}{a} \Delta U, \quad U = \mathcal{H}f \Delta^{-1} \delta, \quad \mathbf{v} = -\mathcal{H}f \nabla \Delta^{-1} \delta. \quad (2.33)$$

In Fourier space, the inverse Laplacian can be readily manipulated, and the velocity vector becomes

$$U(k) = -\frac{\mathcal{H}f}{k^2} \delta(k), \quad \mathbf{v}(k) = i k \frac{\mathcal{H}f}{k^2} \delta(k), \quad (2.34)$$

where we suppressed the time-dependence, for example,

$$\delta(k) = D(t) \delta(k, t_0). \quad (2.35)$$

### 2.2.3 Two-Point Correlation of the Peculiar Velocities

Given the peculiar velocity (vector) field, we can compute the two-point correlation function of the peculiar velocities at two different points:

$$\Psi_{ij}(r) = \langle v_i(x) v_j(x + r) \rangle = \int \frac{d^3k}{(2\pi)^3} e^{i k \cdot r} \mathcal{H}^2 f^2 P_m(k) \frac{k_i k_j}{k^4} \equiv \Psi_\perp(r)(\delta_{ij} - \hat{r}_i \hat{r}_j) + \hat{r}_i \hat{r}_j \Psi_{ij}(r), \quad (2.36)$$

where $\hat{r}_i = r_i / |r|$, the matter density power spectrum is

$$\langle \delta(k) \delta(k') \rangle = (2\pi)^3 \delta_D(k + k') P_m(k), \quad (2.37)$$

and we defined two velocity correlation functions, $\Psi_\parallel$ along the connecting direction and $\Psi_\perp$ perpendicular to it:

$$\Psi_\perp := \int \frac{dk}{2\pi^2} \mathcal{H}^2 f^2 P(k) \frac{j_1(kr)}{kr}, \quad \Psi_\parallel := \int \frac{dk}{2\pi^2} \mathcal{H}^2 f^2 P(k) \left[ j_0(kr) - \frac{2j_1(kr)}{kr} \right] = \frac{d}{dr} [r \Psi_\perp(r)], \quad (2.38)$$

where we used

$$\int d\mu e^{\pm i \mu x} = 2j_0(x), \quad \int d\mu \mu^2 e^{\pm i \mu x} = 2j_0(x) - \frac{4j_1(x)}{x}. \quad (2.39)$$

If we define the multipole correlation function of the matter as

$$\xi^n(x) := \int \frac{dk}{2\pi^2} k^n j_i(kx) P_m(k), \quad (2.40)$$

we can show that the velocity correlation functions are

$$\Psi_\parallel \propto \frac{1}{3} (\xi_0^0 - 2\xi_2^0), \quad \Psi_\perp \propto \frac{1}{3} (\xi_0^0 + \xi_2^0). \quad (2.41)$$

The two-point correlation function of the velocity inner product is then

$$\langle \mathbf{v}(x) \cdot \mathbf{v}(x + r) \rangle = \Psi_\parallel(r) + 2\Psi_\perp(r), \quad (2.42)$$

and its variance is

$$\sigma^2_{3D} \equiv \langle \mathbf{v}(x) \cdot \mathbf{v}(x) \rangle = \int \frac{dk}{2\pi^2} \mathcal{H}^2 f^2 P(k). \quad (2.43)$$

Since the peculiar velocity is often measured along the line-of-sight direction only, one-dimensional variance is often used in literature:

$$\sigma^2_{1D} = \frac{1}{3} \sigma^2_{3D}. \quad (2.44)$$

For the same reason, the two-point correlation function of the line-of-sight velocities is often measured, and it is related to the velocity correlation $\Psi_{ij}$ as

$$\langle V_1 V_2 \rangle = \hat{n}_1 \hat{n}_2 \Psi_{ij}, \quad V_i := \hat{n}_i v_i(x_1), \quad \hat{n}_1 = x_1 / |x_1|, \quad (2.45)$$

where $\hat{n}_1$ is the line-of-sight direction for the position $x_1$. 
2.3 Redshift-Space Distortion

2.3.1 Redshift-Space Power Spectrum

In cosmology, we rarely know the physical distance to any of the cosmological objects, but we can measure their redshift $z$ with relative ease. The redshift-space distance $s$ is then assigned to the object as

$$s = \int_0^z \frac{dz'}{H}.$$  \hspace{1cm} (2.46)

As we discussed in Section 2.2.1, the observed redshift is the sum of the Hubble expansion and the peculiar velocity. However, since it is measured in terms of wavelength, it is more convenient to express it as

$$1 + z \equiv (1 + \bar{z}) (1 + \delta z), \quad z = \bar{z} + (1 + \bar{z}) \delta z,$$  \hspace{1cm} (2.47)

where the redshift $\bar{z}$ in the background would represent the comoving distance to the object in the background

$$r = \int_0^{\bar{z}} \frac{dz'}{H}, \quad d = \frac{r}{1 + \bar{z}},$$  \hspace{1cm} (2.48)

and the peculiar velocity or any contributions to the observed redshift other than the Hubble expansion is described by the perturbation $\delta z$:

$$\delta z = v_p + \cdots.$$  \hspace{1cm} (2.49)

To the linear order in perturbations, we can expand the redshift-space distance as

$$s \simeq r + \frac{1 + z}{H} \delta z = r + \mathcal{V}, \quad \mathcal{V} := \frac{v_p}{H} = -f \frac{\partial}{\partial r} \Delta^{-1} \delta,$$  \hspace{1cm} (2.50)

where we replaced $\bar{z}$ with $z$ at the linear order. Despite the distortion in the radial distance, the number of galaxies we measure in a given area of the sky remains unaffected:

$$n_g(s) d^3s = n_g(r) d^3r.$$  \hspace{1cm} (2.51)

This relation is exact but assumes that the redshift-space distortion is purely radial, ignoring angular displacements.

One can make a progress by expanding equation (2.51) to the linear order in perturbations, and the redshift-space galaxy fluctuation is then

$$\delta s = \delta g - \left( \frac{d}{dr} + \frac{\alpha}{r} \right) \mathcal{V},$$  \hspace{1cm} (2.52)

where the selection function $\alpha$ is defined in terms of the (comoving) mean number density $\bar{n}_g$ of the galaxy sample as

$$\alpha := \frac{d \ln r^2 \bar{n}_g}{d \ln r} = 2 + \frac{r H}{1 + z} \frac{d \ln \bar{n}_g}{d \ln (1 + z)}.$$ \hspace{1cm} (2.53)

By adopting the distant-observer approximation ($r \rightarrow \infty$) and ignoring the velocity contributions, a further simplification can be made:

$$\delta s \simeq \delta g - \frac{d \mathcal{V}}{dr} = \int \frac{d^3k}{(2\pi)^3} e^{i \mathbf{k} \cdot \mathbf{s}} (b + f \mu_k^2) \delta_m(k),$$  \hspace{1cm} (2.54)

where we used the linear bias approximation $\delta_g = b \delta_m$ and the cosine angle between the Fourier mode and the line-of-sight direction is $\mu_k = \hat{s} \cdot \mathbf{k}$. The galaxy power spectrum in redshift-space is then readily computed as

$$P_s(k, \mu_k) = (b + f \mu_k^2)^2 P_m(k).$$  \hspace{1cm} (2.55)

This redshift-space distortion effect was first derived by Nick Kaiser in 1987. Due to our redshift measurements as the radial distance, the Doppler effect affects our observation of the number density in redshift-space, such that the galaxy power spectrum becomes enhanced along the line-of-sight direction, representing the infall toward the overdense region.

- random motion on small scales, growth rate of structure
2.3.2 Multipole Expansion

The Kaiser formula for the redshift-space power spectrum indicates that the power spectrum is anisotropic, i.e., it depends not only a Fourier mode $k$, but also its direction. So, it is often convenient to expand $P_s(k, \mu_k)$ in terms of Legendre polynomials $L_l(x)$ as

$$P_s(k, \mu_k) = \sum_{l=0,2,4} L_l(\mu_k) P^s_l(k) \, ,$$

(2.56)

and the corresponding multipole power spectra are

$$P^s_l(k) = \frac{2l+1}{2} \int_{-1}^{1} d\mu_k \, L_l(\mu_k) P_s(k, \mu_k) \, ,$$

(2.57)

With its simple angular structure, the simple Kaiser formula in equation (2.55) is completely described by three multipole power spectra

$$P^s_0(k) = \left( b^2 + \frac{2fb}{3} + \frac{f^2}{5} \right) P_m(k) \, , \quad P^s_2(k) = \left( \frac{4bf}{3} + \frac{4f^2}{7} \right) P_m(k) \, , \quad P^s_4(k) = \frac{8}{35} f^2 P_m(k) \, ,$$

(2.58)

while any deviation from the linearity or the distant-observer approximation can give rise to higher-order even multipoles ($l > 4$) and deviations of the lowest multipoles from the above equations.

The correlation function in redshift-space is the Fourier transform of the redshift-space power spectrum $P_s(k, \mu_k)$. With the distant-observer approximation the redshift-space correlation function can be computed and decomposed in terms of Legendre polynomials as

$$\xi_s(s, \mu) = \int \frac{d^3k}{(2\pi)^3} \, e^{i\mathbf{k} \cdot \mathbf{s}} \, P_s(k, \mu_k) = \sum_{l=0,2,4} L_l(\mu) \, \xi^s_l(s) \, ,$$

(2.59)

and the multipole correlation functions are related to the multipole power spectra as

$$\xi^s_l(s) = i^l \int \frac{dk \, k^2}{2\pi^2} \, P^s_l(k) j_l(k s) \, ,$$

(2.60)

$$P^s_l(k) = 4\pi (-i)^l \int dx \, x^2 \xi^s_l(x) j_l(k x) \, ,$$

(2.61)

where $j_l(x)$ denotes the spherical Bessel functions and the cosine angle between the line-of-sight direction $\hat{n}$ and the pair separation vector $s$ is $\mu = \hat{n} \cdot \hat{s}$. With the distant-observer approximation, there are no ambiguities associated with how to define the line-of-sight direction of the galaxy pair, as all angular directions are identical.

2.4 Galaxy Clusters

So far, we discussed the two-point statistics of some cosmological probes. One-point statistics such as the number density has also important cosmological information.

2.4.1 Spherical Collapse Model

A simple spherical collapse model was developed long time ago to serve as a toy model for dark matter halo formation. The idea is that a slightly overdense region in a flat universe evolves as if the region were a closed universe, such that it expands almost together with the background universe but eventually turns around and collapses. The overdense region described by the closed universe would collapse to a singularity, but in reality it virializes and stops contracting. By using the analytical solutions for the two universes, we can readily derive many useful relations about the evolution of such overdense regions.
Einstein-de Sitter Universe

A flat homogeneous universe dominated by pressureless matter is called the Einstein-de Sitter Universe:

\[ H^2 = \frac{8\pi G}{3} \rho_m, \quad \rho_m \propto \frac{1}{a^3}. \]  \hspace{1cm} (2.62)

This simple model is indeed a good approximation to the late Universe, before dark energy starts to dominate the energy budget. The evolution equations are

\[ a = \left( \frac{t}{t_0} \right)^{2/3} = \left( \frac{\eta}{\eta_0} \right)^2, \quad \frac{t}{t_0} = \left( \frac{\eta}{\eta_0} \right)^3, \quad \eta_0 = 3t_0, \]  \hspace{1cm} (2.63)

\[ H = \frac{2}{3t}, \quad \mathcal{H} = \frac{2}{\eta}, \quad \rho_m = \frac{1}{6\pi G t^2}, \quad r = \eta_0 - \eta = \frac{2}{H_0} \left( 1 - \frac{1}{\sqrt{1+z}} \right), \]  \hspace{1cm} (2.64)

where the reference point \( t_0 \) satisfies \( a(t_0) = 1 \), but it can be any time \( t_0 \in (0, \infty) \). At a given epoch \( t_0 \), one can define a mass scale

\[ M := \frac{4\pi}{3} \rho_0 = \frac{H_0^2}{2G}, \quad H_0 = \frac{2}{3t_0}, \]  \hspace{1cm} (2.65)

Closed Homogeneous Universe

An analytic solution can be derived for a closed universe with again pressureless matter. The evolution equations for a closed universe are

\[ \frac{\dot{a}}{a} = \frac{1 - \cos \theta}{2}, \quad t = \frac{t_0}{\pi} (\theta - \sin \theta) = \frac{a_0^2 (\theta - \sin \theta)}{2\sqrt{K}}, \quad d\tilde{\eta} = \frac{a_0}{\sqrt{K}} d\theta, \]  \hspace{1cm} (2.66)

\[ \dot{H}^2 = \frac{8\pi G}{3} \rho_m - \frac{K}{a^2} = \frac{K}{a^2} \left( \frac{\dot{a}}{a} - 1 \right), \]  \hspace{1cm} (2.67)

where we used tilde to distinguish quantities in the closed universe from the flat universe and the maximum expansion (or turn-around \( \dot{a}_t \)) is reached at \( \theta = \pi (H_t = 0) \). The density parameters are related to the curvature \( K \) of the universe as

\[ \Omega_m - 1 = -\Omega_k = -\frac{K}{a^2 H_0^2}, \quad K = \frac{8\pi G \rho_0}{3 a_t} = \frac{H_0^2}{a_t} = \frac{2GM}{a_t} = \frac{\pi^2 a_0^2}{4t_0^2}, \]  \hspace{1cm} (2.68)

where \( H_0 \)

Spherical Collapse Model

Matching the density equal at some early time, say \( t_0 \) (i.e., \( \delta_0 = 0 \)), the time evolution of the overdense region can be derived in a non-perturbative way as

\[ 1 + \delta = \frac{\dot{\rho}_m}{\rho_m} = \left( \frac{a}{a_t} \right)^3 = \frac{9}{2} \frac{(\theta - \sin \theta)^2}{(1 - \cos \theta)^3}, \]  \hspace{1cm} (2.69)

where we used

\[ a^3 = \left( \frac{t}{t_0} \right)^2 = \left( \frac{t_0}{\pi t_0} \right)^2 (\theta - \sin \theta)^2, \quad \dot{a}^3 = \left( \frac{\dot{a}_t}{2} \right)^3 (1 - \cos \theta)^3 = \frac{2}{9} \left( \frac{t_0}{\pi t_0} \right)^2 (1 - \cos \theta)^3. \]  \hspace{1cm} (2.70)

Therefore, the density contrast \( \delta_t \) at its maximum expansion

\[ 1 + \delta_t = \frac{9\pi^2}{16} \simeq 5.6, \]  \hspace{1cm} (2.71)

is about a few, while the density contrast \( \delta_v \) at its virialization

\[ 1 + \delta_v = 18\pi^2 \simeq 177.7, \]  \hspace{1cm} (2.72)
**2.4.2 Dark Matter Halo Mass Function**

This simple relation owes to the fact that the spherical collapse model is local in both Eulerian and Lagrangian spaces. If the number density of the objects $X$ is conserved

\[ \rho d^3x = \bar{\rho} d^3q, \quad \rho_X d^3x = \rho_X^L d^3q, \quad \therefore 1 + \delta_X = (1 + \delta)(1 + \delta_X^L), \]

the bias parameters are related as

\[ b_1 = b_1^L + 1, \quad b_2 = b_2^L + \frac{8}{21} b_1^L, \quad b_3 = b_3^L - \frac{13}{7} b_2^L - \frac{796}{1323} b_1^L, \quad b_4 = b_4^L - \frac{40}{7} b_3^L + \frac{7220}{1323} b_2^L + \frac{476320}{305613} b_1^L. \]

This simple relation owes to the fact that the spherical collapse model is local in both Eulerian and Lagrangian spaces.

**2.4.2 Dark Matter Halo Mass Function**

Given the simple spherical collapse model, we would like to associate the collapsed region with some virialized objects like massive galaxy clusters or dark matter halos. Of our main interest is then the number density of such objects in a mass range $M \sim M + dM$, and this is called the mass function.

A simple model called, the excursion set approach, was developed: One starts with a smoothing scale $R$ and its associated mass $M$. The density fluctuation $\delta_R$ after smoothing with $R$ is very small ($\delta_R = 0$, if $R = \infty$), and this region has never reached the critical density threshold $\delta_c$ in its entire history. This implies that there is no virialized object associated with such mass. One then decreases the smoothing scale (or mass), and looks for the collapsed probability: Some overdense regions have at some point in the past reached the critical density, while some underdense regions have never reached the critical density threshold $\delta_c$.

The task of obtaining the mass function boils down to computing the survival probability and expressing it in terms of the multiplicity function. The way to find the survival probability at a given mass scale $M$ is to derive the evolution of

\[ F_c = 1 - \int_{-\infty}^{\delta_c} d\delta P_s = \int_{M}^{\infty} dM \frac{dn}{dM} \frac{M}{\bar{\rho}_m}, \quad \therefore \frac{dn}{dM} = \frac{\bar{\rho}_m}{M} \left( -\frac{\partial F_c}{\partial M} \right) = \frac{\bar{\rho}_m}{M} f(\nu) \frac{d\ln \nu}{dM}, \]

where it is assumed that the mass function only depends on mass and we defined the multiplicity function $f$ through the relation

\[ \nu = \frac{\delta_c(z)}{\sigma(M)}, \quad \int_{0}^{\infty} \frac{d\nu}{\nu} f = 1. \]
the density fluctuation as we decrease the smoothing scale \( R \). The reason is that the region may have already collapsed at a larger mass scale or smoothing scale, and this contribution should be removed in computing the survival probability at a lower mass scale. The survival probability at \( n \)-th step depends on the entire history of the trajectory (non-Markovian process) as

\[
P_s(\delta_n, \sigma_n) d\delta_n = d\delta_n \int_{-\infty}^{\delta_c} d\delta_{n-1} \cdots \int_{-\infty}^{\delta_c} d\delta_1 P_s(\delta_1, \cdots, \delta_n, \sigma_1, \cdots, \sigma_n),
\]

(2.81)
it is notoriously difficult to solve, even numerically. However, once we assume that the fluctuations are independent at each smoothing and are Gaussian distributed (true only in Fourier space at linear order), the trajectory only depends on the previous step (Markovian process) and the survival probability becomes

\[
P_s(\delta_n, \sigma_n) = \int_{-\infty}^{\delta_c} d\delta_{n-1} P_t(\delta_n, \sigma_n | \delta_{n-1}, \sigma_{n-1}) P_s(\delta_{n-1}, \sigma_{n-1}),
\]

(2.82)

where the transition probability \( P_t \) is nothing but a conditional probability. With the boundary condition \( P_s = 0 \) at \( \delta = \delta_c \), the solution is (derived by Chandrasekhar for other purposes)

\[
P_s = \frac{1}{\sqrt{2\pi \sigma}} e^{\frac{-\delta^2}{2\sigma^2}} - \frac{1}{\sqrt{2\pi \sigma}} e^{\left[-\frac{(2\delta_c - \delta)^2}{2\sigma^2}\right]}.
\]

(2.83)
The survival probability for its simplest case is described by a Gaussian distribution, but the second term reflects that there exist equally likely trajectories around the threshold that have reached the threshold in the past. The collapsed fraction is

\[
F_c = 1 - \frac{1}{2} \text{erf} \left( \frac{\nu_c}{\sqrt{2}} \right) - \frac{1}{2} \text{erf} \left( \frac{\nu_c}{\sqrt{2}} \right) = \text{erfc} \left( \frac{\nu_c}{\sqrt{2}} \right),
\]

(2.84)

and the multiplicity function is

\[
f(\nu) = \sqrt{\frac{2}{\pi}} \nu e^{-\nu^2/2}.
\]

(2.85)

Of course, this model relies on many approximations, and it is not accurate. However, it provides physical intuitions, connecting the complicated formation of galaxy clusters and the dynamical evolution of the matter density fluctuations. In general, numerical \( N \)-body simulations are run, and dark matter halos are identified by using some algorithm such as the friends-of-friends method or its variants to derive the mass function from the simulations.
3 Relativistic Perturbation Theory

3.1 Metric Decomposition and Gauge Transformation

3.1.1 FRW Metric and its Perturbations

We describe the background for a spatially homogeneous and isotropic universe with the FRW metric

\[ ds^2 = g_{\mu \nu} dx^\mu dx^\nu = -a^2(\eta) \, d\eta^2 + a^2(\eta) \, g_{\alpha \beta} \, dx^\alpha dx^\beta , \]  

(3.1)

where \( a(\eta) \) is the scale factor and \( g_{\alpha \beta} \) is the metric tensor for a three-space with a constant spatial curvature \( K = -H_0^2 \, (1 - \Omega_{\text{tot}}) \). We use the Greek indices \( \alpha, \beta, \cdots \) for 3D spatial components and \( \mu, \nu, \cdots \) for 4D spacetime components, respectively. To describe the real (inhomogeneous) universe, we parametrize the perturbations to the homogeneous background metric as

\[ g_{00} := -a^2 \left( 1 + 2A \right) , \quad g_{0\alpha} := -a^2 B_\alpha , \quad g_{\alpha \beta} := a^2 \left( \bar{g}_{\alpha \beta} + 2C_{\alpha \beta} \right) , \]  

(3.2)

where 3-tensor \( A, B_\alpha \) and \( C_{\alpha \beta} \) are perturbation variables and they are based on the 3-metric \( \bar{g}_{\alpha \beta} \). Due to the symmetry of the metric tensor, we have ten components, capturing the deviation from the background. The inverse metric tensor can be obtained by using \( g^{\mu \nu} g_{\mu \nu} = \delta^\nu_\mu \) and expanding to the linear order as

\[ g^{00} = \frac{1}{a^2} \left( -1 + 2A \right) , \quad g^{0\alpha} = -\frac{1}{a^2} B^\alpha , \quad g^{\alpha \beta} = \frac{1}{a^2} \left( \bar{g}^{\alpha \beta} - 2C^{\alpha \beta} \right) . \]  

(3.3)

For later convenience, we also introduce a time-like four-vector, describing the motion of an observer \((-1 = u_\mu u^\mu)\):

\[ u^0 = \frac{1}{a} (1 - A) , \quad u^\alpha := \frac{1}{a} U^\alpha , \quad u_0 = -a (1 + A) , \]  

(3.4)

\[ u_\alpha = a (U_\alpha - B_\alpha) := a v_\alpha := a(-v_\alpha + v^{(v)}_\alpha) , \quad v := U + \beta , \quad v^{(v)}_\alpha = U^{(v)}_\alpha - B^{(v)}_\alpha , \]  

(3.5)

where \( U^\alpha \) is again based on \( g_{\alpha \beta} \).

3.1.2 Scalar-Vector-Tensor Decomposition

Given the splitting of the spatial hypersurface and the symmetry associated with it, we decompose the perturbation variables \((to \, all \, orders)\) as

\[ A := \alpha , \quad B_\alpha := \beta_\alpha + B^{(v)}_\alpha , \quad C_{\alpha \beta} := \varphi \bar{g}_{\alpha \beta} + \gamma_{\alpha \beta} + C^{(v)}_{\alpha \beta} + C^{(t)}_{\alpha \beta} , \quad U^\alpha := -U^\alpha + U^{(v)} , \]  

(3.6)

subject to the transverse and traceless conditions:

\[ B^{(v)}_\alpha |_{\alpha} = 0 , \quad C^{(v)}_\alpha |_{\alpha} = 0 , \quad v^{(v)}_\alpha |_{\alpha} = 0 , \quad C^{(t)}_\alpha |_{\alpha} = 0 , \quad C^{(t)}_{\alpha \beta} |_{\alpha \beta} = 0 , \quad U^{(v)} |_{\alpha} = 0 , \]  

(3.7)

where the vertical bar represents the covariant derivative with respect to the 3-metric \( \bar{g}_{\alpha \beta} \):

\[ X^\alpha |_{\beta} = X^\alpha_{,\beta} + \Gamma^\alpha_\beta_\gamma X^\gamma , \quad X_{\alpha |\beta} = X_{\alpha ,\beta} - \Gamma^\gamma_\alpha_\beta X^\gamma . \]  

(3.8)

This simply implies that the scalar perturbations describe the longitudinal modes and the vector \((v)\) and the tensor \((t)\) perturbations describe the transverse modes. Furthermore, the tensor perturbation is traceless. The decomposed scalar perturbations can be obtained as

\[ \beta = \Delta^{-1} \nabla^\alpha B_\alpha , \quad \gamma = \frac{1}{2} \left( \Delta + \frac{1}{2} R \right)^{-1} \left( 3\Delta^{-1} \nabla^\alpha \nabla^\beta C_{\alpha \beta} - C^\alpha_\alpha \right) , \]  

(3.9)

\[ \varphi = \frac{1}{3} C^\alpha_\alpha - \frac{1}{6} \Delta \left( \Delta + \frac{1}{2} R \right)^{-1} \left( 3\Delta^{-1} \nabla^\alpha \nabla^\beta C_{\alpha \beta} - C^\alpha_\alpha \right) , \]
where $\nabla_\alpha$ is the covariant derivative based on $\tilde{g}_{\alpha\beta}$ (i.e., vertical bar) and $\Delta = \nabla^\alpha \nabla_\alpha$ is the Laplacian operator. The presence of the Ricci scalar ($\bar{R} = 6\bar{K}$) for the three-space indicates that covariant derivatives are non-commutative.

$$\bar{R}_{\alpha\beta\gamma\delta} = 2\bar{K} \tilde{g}_{\alpha[\gamma} \tilde{g}_{\delta]\beta}.$$  (3.10)

The decomposed vector and tensor components are computed in a similar manner as

$$B^{(v)}_\alpha = B_\alpha - \nabla_\alpha \Delta^{-1} \nabla^\beta B_\beta, \quad C^{(v)}_\alpha = \mathbf{2} \left( \Delta + \frac{1}{3} \bar{R} \right)^{-1} \left[ \nabla^\beta C_{\alpha\beta} - \nabla_\alpha \Delta^{-1} \nabla^\delta \nabla_\gamma C_{\delta\gamma} \right],$$  (3.11)

$$C^{(t)}_{\alpha\beta} = C_{\alpha\beta} - \frac{1}{3} C_\gamma \tilde{g}_{\alpha\beta} - \frac{1}{2} \left( \nabla_\alpha \nabla_\beta - \frac{1}{3} \tilde{g}_{\alpha\beta} \Delta \right) \left( \Delta + \frac{1}{2} \bar{R} \right)^{-1} \left[ 3\Delta^{-1} \nabla^\gamma \nabla^\delta C_{\gamma\delta} - C_\gamma \right] - 2\nabla_\alpha \left( \Delta + \frac{1}{3} \bar{R} \right)^{-1} \left[ \nabla^\gamma C_{\beta\gamma} - \nabla_\beta \Delta^{-1} \nabla^\delta \nabla_\gamma C_{\delta\gamma} \right],$$

and they satisfy the transverse condition $B^{(v)}_\alpha |_{\alpha} = C^{(v)}_\alpha |_{\alpha} = C^{(t)}_\alpha |_{\alpha} = 0$ and the traceless condition $C^{(t)}_\alpha |_{\alpha} = 0$.

### 3.1.3 Comparison in Notation Convention

Bardeen convention:

$$A \rightarrow \alpha, \quad B^{(0)} \rightarrow -k\beta, \quad H_L \rightarrow \varphi + \frac{1}{3} \Delta \gamma, \quad H_T \rightarrow -\Delta \gamma,$$  (3.12)

$$B^{(1)} Q^{(1)}_\alpha \rightarrow B_\alpha, \quad H^{(1)}_T Q_\alpha \rightarrow -k C_\alpha, \quad H^{(2)}_T Q_\alpha \rightarrow C_{\alpha\beta}.$$  (3.13)

Weinberg convention:

$$\Phi \rightarrow \alpha_\chi, \quad \Psi \rightarrow -\varphi_\chi, \quad \delta u \rightarrow -a v_\chi, \quad \mathcal{R} \rightarrow \varphi_v, \quad \zeta \rightarrow \varphi_\delta,$$  (3.14)

$$\delta p \rightarrow \delta p - \frac{1}{3a^2} \Delta \Pi, \quad \pi^S := \delta \sigma \rightarrow \frac{\Pi}{a^2}, \quad \pi^V := \frac{1}{2a} \Pi_\alpha, \quad \pi^{T}_{ij} \rightarrow \Pi^{(t)}_{\alpha\beta}.$$  (3.15)

### 3.1.4 Gauge Transformation

The general covariance of general relativity guarantees that any coordinate system can be used to describe the physics and it has to be independent of coordinate systems. This is known as the diffeomorphism symmetry in general relativity. However, when we split the metric into the background and the perturbations around it by choosing a coordinate system, we explicitly change the correspondence of the physical Universe to the background homogeneous and isotropic Universe. Hence, the metric perturbations transform non-trivially (or gauge transform), and the diffeomorphism invariance implies that the physics should be gauge-invariant.

The gauge group of general relativity is the group of diffeomorphisms. A diffeomorphism corresponds to a differentiable coordinate transformation. The coordinate transformation on the manifold $\mathcal{M}$ can be considered as one generated by a smooth vector field $\zeta^\mu$. Given the vector field $\zeta^\mu$, consider the solution of the differential equation

$$\frac{d\chi^\mu(\lambda)}{d\lambda} \bigg|_{P} = \zeta^\mu \left[ x^\nu_\mu(\lambda) \right], \quad \chi^\mu_P(\lambda = 0) = x^\mu_P, \quad \frac{d}{d\lambda} = \zeta^\mu \partial_\mu,$$  (3.16)

defines the parametrized integral curve $x^\mu(\lambda) = \chi^\mu_P(\lambda)$ with the tangent vector $\zeta^\mu(x_P)$ at $P$. Therefore, given the vector field $\zeta^\mu$ on $\mathcal{M}$ we can define an associated coordinate transformation on $\mathcal{M}$ as $x^\mu_P \rightarrow x^\mu_P = \chi^\mu_P(\lambda = 1)$ for any given $P$. Assuming that $\zeta^\mu$ is small one can use the perturbative expansion for the solution of equation to obtain

$$x^\mu_P = \chi^\mu_P(\lambda = 1) = \chi^\mu_P(\lambda = 0) + \frac{d}{d\lambda} \chi^\mu_P \bigg|_{\lambda=0} + \frac{1}{2} \frac{d^2}{d\lambda^2} \chi^\mu_P \bigg|_{\lambda=0} + \cdots = x^\mu_P + \zeta^\mu(x_P) + \frac{1}{2} \zeta^\mu \zeta_\nu + O(\zeta^3) = e^{\zeta^\mu \partial_\mu} x^\mu.$$  (3.17)

This parametrization corresponds to the gauge-transformation with $\zeta^\mu$.

In general, any gauge-transformation of tensor $\mathbf{T}$ for an infinitesimal change $\zeta$ can be expressed in terms of the Lie derivative (valid to all orders of $\mathbf{T}$)

$$\delta_{\zeta} \mathbf{T} := \tilde{\mathbf{T}} - \mathbf{T} = -\mathbf{L}_\zeta \mathbf{T} + O(\zeta^2), \quad \mathbf{L}_\zeta A^\mu = A^\mu_\nu \zeta^\nu - A^\mu_\nu A^\nu, \quad \mathbf{L}_\zeta T_{\mu\nu} = T_{\mu\nu,\rho} \zeta^\rho + T_{\mu\rho} \zeta^\rho_\nu + T_{\rho\nu} \zeta^\rho_\mu.$$  (3.18)
where they are all evaluated at the same coordinate and the derivatives in the Lie derivatives can be replaced with covariant derivatives (Lie derivatives are tensorial). To all orders in $\xi$, we have

$$T(x) = T(x) - \mathcal{L}_\xi T + \frac{1}{2} \mathcal{L}_\xi^2 T + \cdots = \exp\left[-\mathcal{L}_\xi\right] T.$$  

Therefore, the gauge-transformation in perturbation theory is simply

$$\delta_\xi \tilde{T} = 0, \quad \delta_\xi T^{(1)} = -\mathcal{L}_\xi T, \quad \delta_\xi T^{(n)} = -\mathcal{L}_\xi T^{(n-1)},$$  

where we used that $\xi$ is also a perturbation.

In fact, there are two ways of looking at the transformation in perturbation theory. For example, the metric tensor has to transform as a tensor. But once we split it into the background and the perturbations, there exist two ways

$$\delta g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad \delta g = -\mathcal{L}_\xi g, \quad \delta h = \mathcal{L}_\xi h, \quad \delta h = -\mathcal{L}_\xi h - \mathcal{L}_\xi \bar{g},$$

where we suppressed the tensor indices. In (1), the background and the perturbation transform altogether like tensors (at the same coordinates), such that the sum transforms like a tensor. In perturbation theory, we do not use this, because the infinitesimal transformation $\xi$ is always considered as a perturbation. However, for example we can consider some general spatial rotation $\zeta$, such that the background metric also changes.\(^1\)

### 3.1.5 Linear-Order Gauge Transformation

At the linear order, the Lie derivative is trivial, and the the most general coordinate transformation in Eq. (3.17) becomes

$$\tilde{x}^\mu = x^\mu + \xi^\mu, \quad \xi^\mu := (T, L^\alpha), \quad L^\alpha := L^\alpha + L^{(v)\alpha},$$

where we now use $\xi^\mu = \mu^\mu$. The transformation of the metric tensor at the leading order in $\xi$ is then

$$\delta_\xi g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) - g_{\mu\nu}(x) = -\left(\xi_{\mu;\nu} + \xi_{\nu;\mu}\right) = -\mathcal{L}_\xi g_{\mu\nu},$$

where the semi-colon represents the covariant derivative with respect to the full metric $g_{\mu\nu}$. Under the coordinate transformation, the scalar quantities gauge-transform as

$$\tilde{\alpha} = \alpha - \frac{1}{a(T)}, \quad \tilde{\beta} = \beta - T + L', \quad \tilde{\varphi} = \varphi - \mathcal{H}T, \quad \tilde{\gamma} = \gamma - L, \quad \tilde{U} = U - L', \quad \tilde{v} = v - T,$$

the vector metric perturbations gauge-transform as

$$\tilde{B}_\alpha^{(v)} = B_\alpha^{(v)} + L'_\alpha, \quad \tilde{C}_\alpha^{(v)} = C_\alpha^{(v)} - L_\alpha, \quad \tilde{U}_\alpha^{(v)} = U_\alpha^{(v)} + L'_\alpha,$$

and the tensor perturbations are gauge-invariant at the linear order.

Based on the above gauge transformation properties, we can construct linear-order gauge-invariant quantities. The gauge-invariant variables are

$$\varphi_v := \varphi - aHv, \quad \varphi_\chi := \varphi - H\chi, \quad v_\chi := v - \frac{1}{a}\chi, \quad \varphi_\delta := \varphi + \frac{\delta \rho}{3(\rho + p)}, \quad \delta_\alpha := \delta - \frac{\dot{a}}{a}v,$$

$$\alpha_\chi := \alpha - \frac{1}{a}\chi', \quad \delta_\phi \varphi := \delta \phi - \frac{\dot{\chi}}{H} \varphi, \quad \Psi_\alpha^{(v)} := B_\alpha^{(v)} + C_\alpha^{(v)}', \quad v_\alpha^{(v)} := U_\alpha^{(v)} - B_\alpha^{(v)},$$

where $\chi := a(\beta + \gamma')$ is the scalar shear of the normal observer ($n_\alpha = 0$) and it is spatially invariant, transforming as $\tilde{\chi} = \chi - aT$. These gauge-invariant variables ($\alpha_\chi, \varphi_\chi, v_\chi, \Psi_\alpha, v_\alpha^{(v)}$) correspond to $\Phi_A, \Phi_H, v_0^{(0)}, \Psi$, and $v_c$ in Bardeen (1980).

\(^1\)In FRW, we use the spatial metric $\tilde{g}_{\alpha\beta}$ unspecified, implying we can do a further spatial transformation to e.g., spherical coordinate and so on and change the background metric (while it remains covariant). The time component is fixed, otherwise it ruins the FRW symmetry ($g_{00}$ component in the background or different coefficient in time component for example.)
3.1.6 Popular Choices of Gauge Condition

By a suitable choice of coordinates, we can set \( T = L = 0 \), simplifying the metric. For simplicity, we only consider the scalar perturbations in the following two cases.

- **The conformal Newtonian Gauge.**— in which we choose the spatial and the temporal gauge conditions:
  \[
  \tilde{\gamma} = \gamma = 0 \quad \rightarrow \quad L = 0 , \quad \tilde{\beta} = \beta = 0 \quad \rightarrow \quad T = 0 , \quad \chi = 0 .
  \]
  All the gauge modes are fixed, and the metric in this gauge condition is
  \[
  ds^2 = -a^2 \left( 1 + 2\psi \right) d\eta^2 + a^2 \left( 1 + 2\phi \right) g_{\alpha\beta} dx^\alpha dx^\beta ,
  \]
  and the velocity vector is then
  \[
  U = v_{\chi} , \quad v = -\nabla U .
  \]
  The metric and its equations appear more like the Newtonian equations, and hence the name. We will use this gauge condition to illustrate and simplify the problems.

- **Synchronous-Comoving Gauge.**— in which we choose the spatial and the temporal gauge conditions:
  \[
  \tilde{\alpha} = \alpha = 0 \quad \rightarrow \quad (aT)' = 0 , \quad \tilde{\beta} = \beta = 0 \quad \rightarrow \quad T = L' ,
  \]
  such that the metric becomes
  \[
  ds^2 = -a^2 d\eta^2 + a^2 \left( \tilde{g}_{\alpha\beta} + 2C_{\alpha\beta} \right) dx^\alpha dx^\beta .
  \]
  All the metric perturbations in this gauge condition are included in the spatial metric tensor. However, as apparent from the above gauge condition, the gauge freedoms are not completely fixed:
  \[
  T = L' = \frac{1}{a} F(x) , \quad L = \int d\eta \frac{F(x)}{a} + G(x) ,
  \]
  where \( F \) and \( G \) are two arbitrary functions of spatial coordinates. Typically, this issue is resolved by assuming additional condition at the initial epoch
  \[
  v = 0 \quad \rightarrow \quad F(x) = 0 , \quad T = 0 .
  \]
  This condition is indeed the temporal comoving gauge condition, and hence the whole choice is often referred to as the comoving-synchronous gauge (or synchronous-comoving). The comoving gauge is often chosen with a different spatial gauge condition (\( \gamma = 0 \)). Note, however, that the spatial function \( G(x) \) is still left unspecified, and hence \( \gamma \) is a gauge mode, whereas \( U \) for example is physical. Due to this deficiency, we will not use this gauge condition in the following.

The notation convention in Ma and Bertschinger (1995):
\[
 h_{ij} := \hat{k}_i \hat{k}_j h + \left( \hat{k}_i \hat{k}_j - \frac{1}{2} \delta_{ij} \right) 6\eta \rightarrow 2C_{ij} , \quad h \rightarrow 6\varphi + 2\Delta \gamma , \quad \eta \rightarrow -\varphi .
\]

### 3.2 Energy-Momentum Tensor

#### 3.2.1 Formal Definition

We will consider a simple action of the matter sector, in addition to the gravity described by the Einstein-Hilbert action:
\[
 S = \int \sqrt{-g} \, d^4x \left[ \frac{R}{16\pi G} + L_m \right] ,
\]
where the matter Lagrangian includes the cosmological fluids and other matter fields such as scalars and so on. The energy-momentum tensor defined by the action
\[
 T_{\mu\nu} = g_{\mu\nu} L_m - 2 \frac{\delta L_m}{\delta g^{\mu\nu}} , \quad T^{\mu\nu} = g^{\mu\nu} L_m + 2 \frac{\delta L_m}{\delta g_{\mu\nu}} ,
\]
is indeed the conserved current (tensor) of the action under the space-time translation invariance. The Noether theorem says that when there exists a (global) symmetry, there exists a conserved current. The space-time translation invariance is the symmetry of general relativity, and the Noether current associated with this symmetry is the energy-momentum tensor:
\[
 T_{\mu\nu} = 0 .
\]
3.2.2 General Decomposition for Cosmological Fluids

For our purposes, we are not interested in the microscopic states of the systems, but interested in their macroscopic states, often described by the density, the pressure, the temperature, and so on. The energy-momentum tensor for a fluid can be expressed in terms of the fluid quantities measured by an observer with four velocity \( u^\mu \) as (the most general decomposition)

\[
T_{\mu \nu} := \rho u_\mu u_\nu + p H_{\mu \nu} + q_\mu u_\nu + q_\nu u_\mu + \pi_{\mu \nu} ,
\]

where \( H_{\mu \nu} \) is the projection tensor and

\[
H_{\mu \nu} = g_{\mu \nu} + u_\mu u_\nu , \quad H_\mu^\mu = 3 , \quad u^\mu q_\mu = 0 = u^\mu \pi_{\mu \mu} , \quad \pi_{\mu \nu} = \pi_{\nu \mu} , \quad \pi_\mu^\mu = 0 . \tag{3.39}
\]

The variables \( \rho, p, q_\mu, \) and \( \pi_{\mu \nu} \) are the energy density, the isotropic pressure (including the entropic one), the (spatial) energy flux and the anisotropic pressure measured by the observer with \( u_\mu \), respectively, i.e.,

\[
\rho = T_{\mu \nu} u^\mu u^\nu , \quad p = \frac{1}{3} T_{\mu \nu} H^{\mu \nu} , \quad q_\mu = - T_{\rho \sigma} u^\rho H_\sigma^\nu , \quad \pi_{\mu \nu} = T_{\rho \sigma} H_\sigma^\rho H_\nu^\mu - p H_{\mu \nu} . \tag{3.40}
\]

Remember that these fluid quantities are observer-dependent. We decompose the fluid quantities into the background and the perturbations:

\[
\rho := \bar{\rho} + \delta \rho , \quad p := \bar{p} + \delta p , \quad \delta p := c_s^2 \delta \rho + e , \quad c_s^2 := \frac{\dot{\bar{p}}}{\bar{\rho}} , \tag{3.41}
\]

\[
q_\alpha := a Q_\alpha , \quad \pi_{\alpha \beta} := a^2 \Pi_{\alpha \beta} , \quad e := \dot{\bar{p}} \Gamma , \quad \Gamma = \frac{\delta p}{\bar{p}} - \frac{\delta \rho}{\bar{\rho}} . \tag{3.42}
\]

where \( Q_\alpha \) and \( \Pi_{\alpha \beta} \) are based on \( \bar{g}_{\alpha \beta} \). At the background level, all the above fluid quantities vanish, except \( \rho = \bar{\rho} \) and \( p = \bar{p} \). For the adiabatic perturbations, we have \( e = 0 \) and the sound speed is \( c_s^2 = \delta p / \delta \rho = \bar{p} / \dot{\bar{\rho}} \). For multiple fluids, we can add up the individual energy-momentum tensor to derive the total energy-momentum tensor. In the case of multiple fluids, their fluid velocities are not necessarily identical, and there exist non-vanishing energy flux. Non-vanishing \( e \) and \( \Gamma \) parametrize the entropic perturbations of the fluids.

Though these relations are exact, we will be concerned with linear-order perturbations. Raising the index of the energy momentum tensor, we derive

\[
T_0^0 = - \rho + \mathcal{O}(2) , \quad T_\alpha^\alpha = (\bar{\rho} + \bar{p}) (U_\alpha - B_\alpha) + Q_\alpha + \mathcal{O}(2) ,
\]

\[
T_\beta^\beta = p \delta_\beta^\beta + \Pi_\beta^\beta + \mathcal{O}(2) , \quad T_0^\alpha = - (\bar{\rho} + \bar{p}) U^\alpha - Q^\alpha ,
\]

and the (spatial) energy flux and the anisotropic pressure satisfy

\[
q_0 = 0 + \mathcal{O}(2) , \quad \pi_{0 \mu} = 0 + \mathcal{O}(2) . \tag{3.45}
\]

3.2.3 Tetrad Approach

Given the observer with \( u^\mu \), one can define a local Lorentz frame (where the metric is Minkowski) by constructing three spacelike orthonormal vectors \([e_1]^\mu\). For example, one can construct three rectangular basis vectors \([e_x]^\mu, [e_y]^\mu, [e_z]^\mu\) and of course \([e_t]^\mu = u^\mu = - [e_i]^\mu\), where the component index \( a \) of \([e_a]^\mu\) represents their coordinates. The orthonormality condition and the spacelike normalization is

\[
\eta_{ab} = g_{\mu \nu} [e_a]^\mu [e_b]^\nu \rightarrow \delta_{ij} = g_{\mu \nu} [e_i]^\mu [e_j]^\nu , \quad 0 = g_{\mu \nu} [e_i]^\mu [e_i]^\nu , \quad -1 = g_{\mu \nu} [e_1]^\mu [e_1]^\nu , \tag{3.46}
\]

where \( a, b, \cdots = t, x, y, z \) represent the local Lorentz indices.

Assuming that the four velocity of the observer is indeed the fluid velocity, we can go to the rest-frame of the fluid by using the tetrad expressions as

\[
\rho = T_{\mu \nu} [e_1]^\mu [e_1]^\nu , \quad p = \frac{1}{3} T_{\mu \nu} \sum_{i=1}^{3} [e_i]^\mu [e_i]^\nu , \quad H_{\mu \nu} = \sum_{i=1}^{3} [e_i]_{\mu \nu} [e_i]^\nu . \tag{3.47}
\]
The other fluid quantities can be readily expressed in terms of the tetrad basis, and the energy-momentum tensor in the rest-frame of the fluid is then

\[ T_{ab} = \begin{pmatrix}
\rho & -q_x & -q_y & -q_z \\
-q_x & p + \pi_{xx} & \pi_{xy} & \pi_{xz} \\
-q_y & \pi_{yx} & p + \pi_{yy} & \pi_{yz} \\
-q_z & \pi_{zx} & \pi_{zy} & p + \pi_{zz}
\end{pmatrix} . \tag{3.48}

The orthogonality condition for the energy-flux and the anisotropic stress implies

\[ 0 = u^\mu q_\mu = q_t , \quad 0 = u^\mu \pi_{\mu
u} = \pi_{ta} , \quad 0 = \pi_\mu^\mu = \pi_t^t + \pi_i^i = \pi_{ii} . \tag{3.49}\]

However, we should pay attention to the difference in the quantities expressed in the rest-frame and in the FRW coordinate:

\[ q_i := q_\mu [e_i]^\mu = Q^{(1)}_\alpha \delta_i^\alpha + \mathcal{O}(2) , \quad \pi_{ij} := \pi_{\mu
u} [e_i]^\mu [e_j]^\nu = \Pi^{(1)}_{\alpha\beta} \delta_i^\alpha \delta_j^\beta + \mathcal{O}(2) . \tag{3.50}\]

As noted, the fluid quantities are observer-dependent. Given the energy momentum tensor in terms of the fluid rest-frame quantities \((q_\mu = 0)\), if the observer is moving with \(e_\mu^i\) relative to the fluid velocity \(u^\mu\), the observer measures different fluid quantities from those defined in the rest frame. The energy-momentum tensor can be projected into the observer rest-frame as

\[ T_{ab} = (\rho + p) u_a u_b + p \eta_{ab} + \pi_{ab} , \quad u_a := u_\mu e_\mu^a , \quad \mathcal{H}_{ij} := \mathcal{H}_{\mu
u} e_i^\mu e_j^\nu = \delta_{ij} , \tag{3.51}\]

where the fluid velocity and the anisotropic pressure satisfy

\[ -1 = u^\mu u_\mu = -u_i^2 + u_t^2 , \quad 0 = \pi_\mu^\mu \rightarrow \pi_{tt} = \pi_{ii} . \tag{3.52}\]

Furthermore, the anisotropic pressure is perpendicular to the fluid velocity:

\[ 0 = u^a \pi_{ab} = u^t \pi_{ta} + u^i \pi_{ia} , \quad \pi_{ti} = \pi_{ij} \frac{u_j}{u_t} . \tag{3.53}\]

Therefore, the energy density and the pressure measured by the observer are

\[ \tilde{\rho} = (\rho + p) u_t^2 - p + \pi_{tt} , \quad \tilde{p} = \frac{1}{3} (\rho + p) u_i u_i + p + \frac{1}{3} \pi_{ii} = p + \frac{1}{3} [(\rho + p) (u_t^2 - 1) + \pi_{tt}] , \tag{3.54}\]

and the anisotropic pressure is

\[ \tilde{\pi}_{ij} = (\rho + p) u_i u_j + \pi_{ij} - \frac{1}{3} \delta_{ij} [(\rho + p) (u_t^2 - 1) + \pi_{tt}] , \quad 0 = \tilde{\pi}_{ti} = \tilde{\pi}_{tt} . \tag{3.55}\]

Since the velocities of the fluid and the observer are different, the observer measures the non-vanishing spatial energy flux

\[ \tilde{q}_i = T_i^t = (\rho + p) u_i u_t + \pi_i^t , \tag{3.56}\]

At the linear order in perturbations, the velocity of the fluid measured by the observer is

\[ u^\mu = e_\mu^a u_a = (1, U^i - U_{\text{obs}}^i) + \mathcal{O}(2) . \tag{3.57}\]

Therefore, the fluid quantities measured by the observer are

\[ 0 = \pi_{tt} = \pi_{ti} , \quad \tilde{\rho} = \rho , \quad \tilde{p} = p , \quad \tilde{\pi}_{ij} = \pi_{ij} , \quad \tilde{q}_i = (\tilde{\rho} + \tilde{p}) (U^i - U_{\text{obs}}^i) . \tag{3.58}\]

In other words, whoever the observer is, the fluid quantities the observer measures are identical to those at the fluid rest frame, except the spatial energy flux.
3.2.4 Distribution Function

In cosmology, photons and neutrinos are the most important radiation components, and they are not described by the fluid approximation. Their statistical properties are captured by the distribution function $F$:

$$F := \bar{f} + f,$$  

(3.59)

where the background distribution $\bar{f}$ often follows the equilibrium distribution and the perturbation $f$ describes the deviation from the equilibrium. The equilibrium distribution for massless particles is fully described by the physical momentum and the temperature, and it is independent of position and time.\(^2\) In the rest-frame of an observer, the physical energy $E$ and the momentum $P^\alpha$ can be measured, and the energy-momentum tensor can be re-constructed as

$$T^{ab} = g \int \frac{d^3 P}{E} P^a P^b F,$$  

(3.60)

where the four momentum satisfies the on-shell condition $-m^2 = P^\alpha P_\alpha$ and $E = P^t$ and $g$ is the spin-degeneracy of the particle, equal to two for photons and one for left-handed neutrinos.

The fluid elements can be readily computed as

$$\rho = g \int d^3 P \bar{f} E f,$$  

$$\mathbf{q}^i = Q_\alpha j_i^\alpha = g \int d^3 P P^i F,$$  

$$\mathbf{p} \delta^{ij} + \pi^{ij} = g \int d^3 P \frac{P^i P^j}{E} F.$$  

(3.61)

For later convenience, we introduce an angular decomposition

$$f(P, \hat{n}) := \sum_{lm} (-i)^l \sqrt{\frac{4\pi}{2l+1}} f_{lm}(P) Y_{lm}(\hat{n}),$$  

$$f_{lm}(P) \equiv (-i)^l \sqrt{\frac{2l+1}{4\pi}} \int d^2 \hat{n} Y^*_{lm}(\hat{n}) f(P, \hat{n}),$$  

(3.62)

where $P^i = P n^i$ and $\hat{n}$ is the unit directional vector. The normalization convention may differ in literature. The perturbations in the fluid quantities are then related to the distribution function as

$$\delta \rho = 4\pi g \int_0^\infty dP P^2 E f_{00},$$  

$$\delta p = \text{Tr} \delta T^{ij} = \frac{4\pi g}{3} \int_0^\infty dP \frac{P^4}{E} f_{00},$$  

(3.63)

where we performed the angular integration. Higher moments of the fluid elements will be related to the higher-moments of the distribution function.

At the linear order, the spatial energy flux from the distribution function is related to the relative velocity as

$$q_i = (\bar{\rho} + \bar{\rho}) (U^i_f - U^i_{\text{obs}}) = Q_\alpha j_i^\alpha,$$  

(3.64)

and the off-diagonal part of the energy-momentum tensor is

$$T^0_\alpha = (\bar{\rho} + \bar{\rho}) (U^{\text{obs}}_{\alpha} - B_{\alpha}) + Q_\alpha = (\bar{\rho} + \bar{\rho}) (U^f_{\alpha} - B_{\alpha}).$$  

(3.65)

3.3 Einstein Equations

3.3.1 Christoffel Symbols

In the absence of the torsion, the Christoffel symbols are uniquely determined by the metric tensor as

$$\Gamma^\mu_{\nu\rho} = \frac{\partial x^\mu}{\partial x^\rho} \frac{\partial^2 \xi^\sigma}{\partial x^\nu \partial x^\rho} = \frac{1}{2} g^{\mu\sigma} (g_{\nu\sigma,\rho} + g_{\rho\sigma,\nu} - g_{\nu\rho,\sigma}),$$  

$$0 = \frac{d^2 \xi^\mu}{d\tau^2},$$  

$$d\tau^2 = -\eta_{\mu\nu} d\xi^\mu d\xi^\nu,$$  

(3.66)

\(^2\)In the background, the physical momentum and the temperature redshift in the same way.
where $\xi^\mu$ is a freely falling coordinate and $d\tau$ is the proper time. To the linear order in perturbations, we derive

$$
\Gamma^\alpha_{\beta\gamma} = \frac{a'}{a} A' \rightarrow \mathcal{H} + \psi', \quad \Gamma^0_{\alpha\beta} = A_{,\alpha} - \frac{a'}{a} B_{\alpha} \rightarrow \psi_{,\alpha},
$$

(3.67)

$$
\Gamma^\alpha_{00} = A^\alpha - B^\alpha - \frac{a'}{a} B^\alpha \rightarrow \psi^\alpha,
$$

(3.68)

$$
\Gamma^0_{\alpha\beta} = \frac{a'}{a} g_{\alpha\beta} - 2 \frac{a'}{a} g_{\alpha\beta} A + B_{(\alpha} |_{\beta) + C_{\alpha\beta} + 2 \frac{a'}{a} C_{\alpha\beta} \rightarrow \mathcal{H} g_{\alpha\beta} (1 - 2 \psi) + (\phi' + 2 \mathcal{H} \phi) g_{\alpha\beta},
$$

(3.69)

$$
\Gamma^\alpha_{0\beta} = \frac{a'}{a} \delta^\alpha_{\beta} + \frac{1}{2} \left( B_{\beta} |_{\alpha} - B_{\alpha} |_{\beta} \right) + C_{\alpha\beta} \rightarrow \mathcal{H} \delta^\alpha_{\beta} + \phi' \delta^\alpha_{\beta},
$$

(3.70)

$$
\Gamma^\alpha_{\beta\gamma} = \Gamma_{\beta\gamma} + \frac{a'}{a} g_{\beta\gamma} B^\alpha + 2 C_{(\beta|\gamma)} - C_{\beta\gamma} |^\alpha \rightarrow \Gamma_{\beta\gamma} + 2 \phi' \delta_{\beta\gamma} - \phi' \delta_{\beta\gamma},
$$

(3.71)

where the conformal Hubble parameter is $\mathcal{H} = a'/a$ and $\Gamma^\alpha_{\beta\gamma}$ is the Christoffel symbols based on 3-metric $\bar{g}_{\alpha\beta}$.

### 3.3.2 Riemann Tensor

The Riemann tensor can then be constructed in terms of the Christoffel symbols as

$$
R_{\mu\nu|\rho\sigma} := \Gamma^\sigma_{\mu\sigma,\rho} - \Gamma^\sigma_{\nu\rho,\sigma} + \Gamma^\rho_{\mu\nu} \Gamma^\sigma_{\rho\sigma} - \Gamma^\rho_{\nu\sigma,\rho} - \Gamma^\sigma_{\mu\sigma,\rho} + \Gamma^\rho_{\nu\sigma} \Gamma^\sigma_{\rho\sigma},
$$

(3.72)

and the Riemann tensor has all the information of the geometry, such that how any four vector changes locally is fully determined by the Riemann tensor

$$
2 u_{\mu;[\nu\rho]} = u_{\sigma} R^\sigma_{\mu\nu\rho}.
$$

(3.73)

Out of the Riemann tensor, we can construct the Ricci tensor (and Ricci scalar) by contracting the Riemann tensor as

$$
R_{\mu\nu} := R^\sigma_{\mu\nu\sigma}, \quad R = R^\mu_{\mu},
$$

(3.74)

and construct the (conformal) Weyl tensor as

$$
C_{\mu\nu|\rho\sigma} := R_{\mu\nu|\rho\sigma} - \frac{1}{2} \left( g_{\mu\rho} R_{\nu\sigma} + g_{\nu\sigma} R_{\mu\rho} - g_{\nu\rho} R_{\mu\sigma} - g_{\mu\sigma} R_{\nu\rho} \right) + \frac{R}{6} \left( g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right).
$$

(3.75)

The Riemann tensor has the symmetry:

$$
R_{\mu\nu\rho\sigma} = R_{[\mu\nu][\rho\sigma]} = R_{\rho\sigma\mu\nu}, \quad R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} = R_{[\mu\nu][\rho\sigma]} = 0,
$$

(3.76)

such that its 20 independent components can be separated into the Ricci tensor (10 components) and the (traceless) Weyl curvature tensor (also 10 components with all the symmetry of the Riemann tensor)

$$
C^\mu_{\mu\nu\sigma} = 0, \quad C^\mu_{\mu\sigma\nu} = C^\mu_{[\mu][\nu\sigma]} = C^\mu_{[\rho][\nu\sigma]} = 0.
$$

(3.77)

The Ricci tensor is algebraically set by matter distribution through the Einstein equation, but the Weyl tensor is determined by differential equations with suitable boundary conditions.

To the background, we derive

$$
\bar{R}^\alpha_{\beta\gamma\delta} = 2 K \delta^\alpha_{[\gamma} \bar{g}_{\delta]\beta}, \quad R^\alpha_{\beta\gamma\delta} = 2 \left( K + \mathcal{H} \right) \delta^\alpha_{[\gamma} \bar{g}_{\delta]\beta}, \quad C_{\alpha\beta}[|\gamma\delta] = K \left( \bar{g}_{\alpha[\delta} C_{\gamma]\beta] + \bar{g}_{\beta[\delta} C_{\gamma]\alpha] \right),
$$

(3.78)

where $\bar{R}^\alpha_{\beta\gamma\delta}$ is the Riemann tensor based on 3-metric $\bar{g}_{\alpha\beta}$. To the linear order in perturbations, we derive the Riemann
The Einstein equation is that the Einstein tensor yields the Friedmann equation gives
\[
\rho \frac{dc}{dt} = \frac{\Lambda}{8\pi G} ,
\]
and the Ricci tensor is completely set by the trace of the energy-momentum tensor. To the background, the Einstein equation yields the Friedmann equation give
\[
H^2 = \frac{8\pi G}{3} \rho - \frac{K}{a^2} + \frac{\Lambda}{3} ,
\]
and the Ricci tensor is completely set by the trace of the energy-momentum tensor. To the background, the Einstein equation yields the Friedmann equation give
\[
H^2 + \dot{H} = \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3p) + \frac{\Lambda}{3} ,
\]
and the energy-momentum conservation yields

\[ \dot{\rho} + 3H (\rho + p) = 0 \, . \]  

(3.94)

In terms of the conformal time, the Friedmann equation becomes

\[ \mathcal{H} = \frac{a^2}{H^2 + \dot{H}} = -\frac{4\pi G}{3} a^2 (\rho + 3p) \, , \quad \mathcal{H}^2 + \mathcal{H}' = \frac{a''}{a} + 4\pi G \frac{a^2}{3} (\rho - 3p) - K \, . \]  

(3.95)

In a flat Universe \((K = 0)\) dominated by an energy component \(\rho \propto a^{-n}\), we can derive the analytic solutions to the Friedmann equation:

\[ a \propto \eta^{-\frac{3}{n-2}} \propto t^{2/n} \, , \quad t \propto \eta^n \, , \quad H = H_o \left( \frac{t_o}{t} \right) = \frac{2}{nt} \, , \quad \mathcal{H} = \mathcal{H}_o \left( \frac{\eta_o}{\eta} \right) = \frac{2}{n} - \frac{1}{\eta} \, . \]  

(3.96)

### 3.3.4 Linear-Order Einstein Equation

The Einstein equation can be expanded up to the linear order in perturbations, and decomposed into the evolution equations describing the scalar, the vector, and the tensor perturbations. At the linear order, they do not mix.

- **Scalar perturbations.**

  \[ G_0^0 : \quad H \kappa + \frac{\Delta + 3K}{a^2} \varphi = -4\pi G \delta \rho \, , \]  

  (3.97)

  \[ C_0^\alpha : \quad \kappa + \frac{\Delta + 3K}{a^2} \chi = 12\pi G (\rho + p) a v \, , \]  

  (3.98)

  \[ G_\alpha - G_0^0 : \quad \kappa + 2H \kappa + \left( 3\dot{H} + \frac{\Delta}{a^2} \right) \alpha = 4\pi G (\delta \rho + 3\delta p) \, , \]  

  (3.99)

  \[ G_\alpha^\beta - \frac{1}{3} \delta_\alpha^\beta G_\gamma^\gamma : \quad \dot{\chi} + H \chi - \varphi - \alpha = -8\pi G \Pi \, , \]  

  (3.100)

  where we defined

  \[ \kappa = 3H \alpha - 3\dot{\varphi} - \frac{\Delta}{a^2} \chi \, , \quad \chi := a \beta + a \gamma' \, , \quad \Pi_{\alpha\beta} := \frac{1}{a^2} \left( \Pi_{\alpha\beta} - \frac{1}{3} \delta_{\alpha\beta} \Delta \Pi \right) + \frac{1}{a} \Pi_{(\alpha|\beta)} + \Pi^{(t)}_{\alpha\beta} \, . \]  

(3.101)

The energy-momentum conservation yields

\[ T_{0\nu}^\nu : \quad \dot{\rho} + 3H (\delta \rho + \delta p) - (\rho + p) \left( \kappa - 3H \alpha + \frac{1}{a} \Delta \nu \right) = 0 \, , \]  

(3.102)

\[ T_{\alpha\nu}^\nu : \quad \frac{a^4 (\rho + p) v_{\alpha}}{a^4 (\rho + p)} - \frac{1}{a} \alpha - \frac{1}{a (\rho + p)} \left( \delta \rho + \frac{2}{3} \frac{\Delta + 3K}{a^2} \Pi \right) = 0 \, . \]  

(3.103)

- **Vector perturbations.**

  \[ G_0^\alpha : \quad \frac{\Delta + 2K}{2a^2} \Psi^{(v)}_\alpha + 8\pi G (\rho + p) v^{(v)}_\alpha = 0 \, , \]  

  (3.104)

  \[ C_0^\alpha : \quad \dot{\Psi}^{(v)}_\alpha + 2H \Psi^{(v)}_\alpha = 8\pi G \Pi^{(v)}_\alpha \, , \]  

  (3.105)

  \[ T_{\alpha\nu}^\nu : \quad \frac{a^4 (\rho + p) v^{(v)}_\alpha}{a^4 (\rho + p)} + \frac{\Delta + 2K}{2a^2} \frac{\Pi^{(v)}_\alpha}{\rho + p} = 0 \, . \]  

(3.106)

- **Tensor perturbations.**

  \[ G_{\alpha\beta} : \quad \ddot{C}_{\alpha\beta}^{(t)} + 3H \dot{C}_{\alpha\beta}^{(t)} - \frac{\Delta - 2K}{a^2} C_{\alpha\beta}^{(t)} = 8\pi G \Pi_{\alpha\beta}^{(t)} \, . \]  

(3.107)
3.3.5 Einstein Equation in the conformal Newtonian Gauge

In the conformal Newtonian gauge we have

\[ \kappa = 3H\psi - 3\dot{\phi}, \quad \chi = 0, \quad U = v\chi, \quad v = -\nabla U. \]  \hfill (3.108)

By substituting into the Einstein equation, we derive

\[ H\kappa + \frac{\Delta + 3K}{a^2} \phi = -4\pi G\delta\rho \rightarrow (\Delta + 3K) \phi + a\dot{\phi} \left(3H\psi - 3\dot{\phi}\right) = -4\pi Ga^2\delta\rho, \]  \hfill (3.109)

\[ \kappa = 12\pi G(\rho + p)av, \quad \phi + \psi = -8\pi G\Pi, \]  \hfill (3.110)

\[ \ddot{\kappa} + 2H\kappa + \left(3\dot{H} + \frac{\Delta}{a^2}\right) \psi = 4\pi G (\delta\rho + 3\delta\rho), \]  \hfill (3.111)

Equation (3.109) can be further manipulated as

\[ (k^2 - 3K)\phi = 4\pi Ga^2\bar{\rho} \delta + aH\kappa = 4\pi Ga^2\bar{\rho} [\delta + 3Hv(1 + w)]. \]  \hfill (3.112)

Assuming a flat Universe \((K = 0)\) and a pressureless medium \((p = \delta p = 0)\), we can further simplify the equation as

\[ \phi + \psi = 0, \quad \kappa = \frac{3}{a} (a\psi)' = 12\pi G\rho av, \quad \Delta \phi = -4\pi Ga^2\rho \delta_v, \quad v' + Hv = \psi, \]  \hfill (3.113)

where \(\delta_v\) is the density fluctuation in the comoving gauge:

\[ \delta_v = \delta + 3Hv. \]  \hfill (3.114)

As we will see, the comoving-gauge curvature perturbation \(\varphi_v\) is conserved on large scales throughout the evolution. So, it is convenient to derive the relation of the conformal Newtonian gauge quantities to the comoving-gauge curvature (sometimes denoted as \(\zeta\) or \(R\))

\[ \varphi_v = \phi - Hv = \phi - \frac{H \left(3H\psi - 3\dot{\phi}\right)}{12\pi Ga(\rho + p)} = \phi - \frac{2\psi - \dot{\phi}/H}{3(1 + w)} \rightarrow_{k=0} \frac{5 + 3w}{3(1 + w)} \phi, \]  \hfill (3.115)

where we assume \(\Pi = 0\) and \(\dot{\phi} = 0\) in the super horizon limit.
4  Standard Inflationary Models

Standard single field inflationary models provide a mechanism for the inflationary expansion (horizon problem) and the perturbation generation (initial condition) by a single scalar field, called inflaton. The scalar field Lagrangian has the canonical kinetic term, but various single field models differ in the scalar field potential, according to which the inflaton rolls over. In most cases, the slow-roll condition is adopted, such that the scalar field dynamics is insensitive to the details of the scalar field potential.

The outcome of the standard model predictions is as follows: The curvature fluctuations are scale-invariant \((n_s \simeq 1)\) and highly Gaussian. The tensor fluctuations are also scale-invariant, but its amplitude is very small compared to the scalar fluctuations. The running of the indices is very small. Recent observations confirm these predictions and constrain the parameters with high precision. However, beyond these basic features/constraints, we do not have a solid model for inflation. Note that the energy scale of inflation is beyond the validity of the standard model physics, and most inflationary models have many theoretical issues, when quantum corrections are considered.

4.1  Single Scalar Field

4.1.1  Scalar Field Action

In addition to the Einstein-Hilbert action for gravity, we consider the action for a scalar field with canonical kinetic term and the potential \(V\):

\[
S = \int \sqrt{-g} \, d^4x \left[ \frac{c^4}{16\pi G} R - \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - V(\phi) \right],
\]

where the kinetic term in the Minkowski spacetime reduces to the standard form

\[
-\frac{1}{2} g^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi = \frac{1}{2} \left( (\partial_t \phi)^2 - (\nabla \phi)^2 \right).
\]

The Euler-Lagrange equation yields the equation of motion for the scalar field

\[
\Box \phi - V,\phi = 0,
\]

and the energy-momentum tensor is

\[
T_{\mu\nu} = g_{\mu\nu} L_\phi - 2 \frac{\delta L_\phi}{\delta g^\mu\nu} \phi,\mu \phi,\nu - \frac{1}{2} g_{\mu\nu} \phi,\rho \phi,\rho - V g_{\mu\nu}.
\]

It is often in literature that the Planck unit is adopted, and there exist two different conventions:

\[
M_{\text{pl}}^2 := \frac{1}{8\pi G}, \quad m_{\text{pl}}^2 := \frac{1}{G}.
\]

4.1.2  Background Relation and Evolution Equations

In the background, the non-vanishing fluid quantities for a scalar field are the energy density and the pressure

\[
\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad p_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi),
\]

and the equation of motion becomes

\[
\ddot{\phi} + 3H \dot{\phi} + V,\phi = 0, \quad \phi'' + 2\mathcal{H} \dot{\phi} + a^2 V,\phi = 0.
\]

The Friedmann equation for a scalar field is

\[
H^2 = \frac{\rho_\phi}{3M_{\text{pl}}^2}, \quad \dot{H} = -\frac{\rho_\phi + p_\phi}{2M_{\text{pl}}^2} = -\frac{\dot{\phi}^2}{2M_{\text{pl}}^2}, \quad H^2 + \dot{H} = \frac{\ddot{a}}{a} = \frac{1}{3M_{\text{pl}}^2} \left( V - \dot{\phi}^2 \right),
\]

where we assumed a flat universe and no cosmological constant. If the potential energy of the scalar field is the dominant energy component of the Universe or the kinetic energy is smaller than the potential energy (slow-roll), the expansion of the Universe is accelerating \(\ddot{a} > 0\). Various inflationary models with slow-roll condition state that the potential is sufficiently flat, such that \(V(\phi)\) is nearly constant during the inflationary period and \(\phi\) slowly evolves (rolls over \(V\)).
4.1.3 de-Sitter Spacetime

The de-Sitter universe is a highly symmetric spacetime, defined as a background FRW universe with no matter and constant Hubble parameter. A constant Hubble parameter leads to an exponential expansion, and we parametrize the de-Sitter solution as

\[ H^2 := \frac{\Lambda}{3}, \quad a(t) = e^{Ht} = -\frac{1}{H\eta}, \quad a = (0, \infty), \quad t = (-\infty, \infty), \quad \eta = (-\infty, 0), \quad (4.9)\]

where the scale factor is normalized at \( t = 0 \). The slow-roll parameter for the de-Sitter spacetime is

\[ \varepsilon := -\frac{\dot{H}}{H^2} = \frac{d}{dt}\left(\frac{1}{H}\right) = 0. \quad (4.10)\]

4.1.4 Slow-Roll Parameters

In general, inflationary models slightly deviate from the de-Sitter phase (\( \varepsilon \neq 0 \)), and its deviation is captured by the slow-roll parameter:

\[ \varepsilon = \frac{d}{dt}\left(\frac{1}{H}\right) = -\frac{\ddot{a}}{aH^2}, \quad \dot{H} = -H^2\varepsilon, \quad \frac{\ddot{a}}{a} = H^2 + \dot{H} = H^2(1 - \varepsilon), \quad (3 - \varepsilon)H^2 = \frac{V}{M_{pl}^2}. \quad (4.11)\]

To solve the horizon problem, we know that the comoving horizon has to decrease in time

\[ 0 > \frac{d}{dt}\left(\frac{1}{H}\right) = -\frac{\ddot{a}}{a^2H^2} = -\frac{1 - \varepsilon}{a}. \quad (4.12)\]

The background evolution of a scalar field can be re-phrased in terms of the slow-roll parameters as

\[ \varepsilon = \frac{1}{2} H^2 M_{pl}^2 = \frac{3}{2}(1 + w), \quad \dot{\phi}^2 = \rho_\phi + p_\phi. \quad (4.13)\]

If we ignore the second derivative of the field (\( \ddot{\phi} \simeq 0 \)) in the equation of motion,

\[ 3H\dot{\phi} \simeq -V_\phi, \quad \rho_\phi + p_\phi \simeq \left(\frac{V_\phi}{3H}\right)^2, \quad (4.14)\]

the slow-roll parameters are then further related to the slow-roll parameters defined in terms of the derivatives of the potential also used below)

\[ \varepsilon_V := \frac{M_{pl}^2}{2} \left(\frac{V_\phi}{V}\right)^2 \simeq \varepsilon, \quad \eta_V := M_{pl}^2 \left(\frac{V_{\phi\phi}}{V}\right) \simeq \varepsilon + \eta, \quad \xi_V := \frac{M_{pl}^4 V_\phi V_{\phi\phi} V}{V^2}, \quad (4.15)\]

where we used the second slow-roll parameter

\[ \eta := -\frac{\ddot{\phi}}{H\dot{\phi}}. \quad (4.16)\]

In fact, one can show the exact relation

\[ \varepsilon = \varepsilon_V \left(1 - \frac{4}{3}\varepsilon_V + \frac{2}{3}\eta_V\right). \quad (4.17)\]

In literature, different convention for slow-roll parameters are often used, in particular, in terms of Hubble flow:

\[ \varepsilon_1 := \varepsilon, \quad \varepsilon_2 := \frac{1}{H}\frac{d\ln \varepsilon}{dt} = 2(\varepsilon - \eta), \quad \varepsilon_{i+1} := \frac{1}{H}\frac{d\ln \varepsilon_i}{dt}. \quad (4.18)\]

Furthermore, the inflation has to last for some time, such that the modes we measure in CMB have to expand at least by \( 40-60 \) e-folds. So it is convenient to define the number of e-folding for a given mode as the number of e-folds the mode \( k \) expanded from the horizon crossing until the end of inflation,\(^\dagger\)

\[ N(\phi_k) := \ln \frac{a_{end}}{a(\phi_k)} = \int_{t_k}^{t_{end}} H \, dt, \quad k = aH. \quad (4.19)\]

\(^\dagger\)The end of inflation is a bit ill-defined, as we do not have a concrete model. However, in terms of \( N \) we can safely use the condition that the slow-roll parameter becomes order unity \( \varepsilon \simeq 1 \).
where \( t_k \) is the time the \( k \)-mode crosses the horizon. Using the \( e \)-folding number, we can express the slow-roll parameters as
\[
dN = H dt = d \ln a, \quad \varepsilon = -\frac{d \ln H}{dN}, \quad \varepsilon_{i+1} = \frac{d \ln \varepsilon_i}{dN}.
\] (4.20)

### 4.1.5 Linear-Order Evolution

Given the energy momentum tensor, we can derive the fluid quantities for a scalar field:
\[
\delta \rho_\phi = \phi \dot{\phi} - \dot{\phi}^2 \alpha + V_\phi \phi \delta \phi = \delta \rho_\phi - 3H \dot{\phi} \delta \phi, \quad \delta \rho_v := \delta \rho - \rho' v,
\]
(4.21)
\[
\delta p_\phi = \phi \ddot{\phi} - \dot{\phi}^2 \alpha - V_\phi \phi \delta \phi = \delta p_\phi - 3c_s^2 H \dot{\phi} \delta \phi, \quad v_\phi = \frac{\delta \phi}{\phi'},
\]
(4.22)
\[
e := \delta p - c_s^2 \delta \rho = (1 - c_s^2) \delta \rho_v, \quad \pi^\alpha_\beta := q^\alpha_\beta = 0,
\]
(4.23)
where we used the following relation and the sound speed is defined as
\[
\dot{\phi} = \dot{\phi} + 3H \dot{\phi} + \left( V_\phi + \frac{k^2}{a^2} \right) \delta \phi = \dot{\phi}(\dot{\alpha} + \kappa) + (2\dot{\phi} + 3H \dot{\phi}) \alpha.
\] (4.26)

Therefore, the comoving gauge corresponds to the uniform field gauge for the single-field models:
\[
\varphi_v = \varphi - Hv = \varphi - H \frac{\dot{\phi}}{\phi} = \varphi_\delta \phi.
\] (4.25)

The equation of motion for a scalar field is then
\[
\delta \ddot{\phi} + 3H \delta \dot{\phi} + \left( V_\phi + \frac{k^2}{a^2} \right) \delta \phi = \dot{\phi}(\dot{\alpha} + \kappa) + (2\dot{\phi} + 3H \dot{\phi}) \alpha.
\] (4.26)

Using the Einstein equations, we derive the governing equation for Mukhanov variable \( \Phi \) (which is the comoving-gauge curvature)
\[
\Phi := \varphi_v - \frac{K/a^2}{4\pi G(\rho + p)} \varphi_\chi = \frac{H^2 a^2}{4\pi G(\rho + p)a} \left( \frac{1}{H} \dot{\varphi}_\chi \right) + \frac{2H^2 \Pi}{\rho + p},
\]
(4.27)
\[
\dot{\Phi} = -\frac{H k^2}{4\pi G(\rho + p)} c_s^2 \dot{\varphi}_\chi - \frac{H}{\rho + p} \left( e - \frac{2k^2}{3a^2} \right) \equiv -\frac{Hc_A^2}{4\pi G(\rho + p)} \frac{k^2}{a^2} \dot{\varphi}_\chi,
\]
(4.28)
where the derivation is fully general and we defined the physical sound speed \( c_A \) for inflaton
\[
c_A^2 := c_s^2 + 4\pi G \frac{a^2}{K} \frac{e}{\dot{\varphi}_\chi} \equiv 1.
\] (4.29)
It is clear that the comoving-gauge curvature is conserved on super horizon scales.

### 4.2 Quantum Fluctuations in Quadratic Action

The background relation describes the inflationary expansion, and the equation of motion we derived describes the evolution of the perturbations at the linear order. Here we will derive their statistical properties. However, before we proceed, we need to better understand the structure of the theory. Even for the standard inflationary models of a single field, the theory is not a free-field, but an interacting field theory.

This can be illustrated as follows. To simplify the calculations, we choose the comoving gauge
\[
0 = v_\phi = \frac{\delta \phi}{\phi'}, \quad \phi(x) = \phi(t), \quad \zeta := \varphi_v = \varphi_\delta \phi,
\] (4.30)
and it coincides with the uniform field gauge. Our main variable for scalar fluctuation is then the comoving gauge curvature \( \zeta \), as the scalar field is uniform. We can expand the action perturbatively to give
\[
S = S_0(\bar{\phi}, g_{ab}) + S_2[\zeta^2] + S_3[\zeta^3] + \cdots, \quad H = H_0 + H_{\text{int}}, \quad H_{\text{int}} = \sum_i F_i(\varepsilon, \eta, \cdots) \zeta^3(\tau) + \cdots,
\] (4.31)
where the background action $S_0$ defines the background evolution and its slow-roll parameters. Here we will study the quadratic action $S_2$ in great detail to derive the power spectrum of the scalar and tensor fluctuations, and the quadratic action is indeed a free-field action in the de-Sitter background (or with small deviations around it). However, remember that the full theory is interacting, and we cannot use the free-field theory to quantize the fluctuations, if we go beyond the quadratic action or compute the high-order correlation functions.

### 4.2.1 Quadratic Action for Scalars

To derive the linear-order equation of motion, we need to expand the action to the quadratic in perturbations. To simplify the calculations, we choose the comoving gauge. After some integrations by part of the quadratic action, the quadratic action becomes

$$S_{(2)} = \frac{1}{2} \int dt \, d^3x \, a^3 \frac{\partial^2}{\partial t^2} \left[ \dot{\zeta}^2 - \frac{1}{a^2} (\nabla \zeta)^2 \right] = \frac{1}{2} \int d\eta \, d^3x \left[ (v')^2 - (\nabla v)^2 + \frac{z''}{z} \right],$$

(4.32)

where we assume $M_{\text{pl}} = 1$ and we defined the canonically-normalized (Mukhanov-Sasaki) variable

$$v := z \zeta, \quad \zeta := \varphi_v, \quad z^2 := a^2 \frac{\partial^2}{\partial t^2} = 2a^2 \varepsilon.$$

(4.33)

The Lagrangian now takes the form of the simple harmonic oscillator, but with time-dependent mass term

$$m^2(\eta) := -\frac{z''}{z}, \quad \frac{a''}{a} = -\frac{2}{\eta^2}.$$  

(4.34)

where we took the de-Sitter limit ($\varepsilon = z = 0$). The canonical momentum and the Hamiltonian are then

$$\pi = \frac{\delta \mathcal{L}}{\delta \dot{v}} = v', \quad \mathcal{H} = \pi v' - \mathcal{L} = \frac{1}{2} \left[ (v')^2 + (\nabla v)^2 + m^2 v^2 \right].$$

(4.35)

The equation of motion for the Mukhanov-Sasaki variable is the Klein-Gordon equation:

$$(\Box - m^2) v = 0, \quad v'' + \omega_k^2 v_k = 0, \quad \omega_k^2 := k^2 + m^2, \quad v_k = v^*_k.$$  

(4.36)

The module functions take the simple solution for the time-dependence under the assumption that $\omega_k \simeq k$ is time-independent in the limit $\eta \to -\infty$:

$$v_k(\eta) \equiv v_k^+ e^{i\omega_k \eta} + v_k^- e^{-i\omega_k \eta} := v_k^+(\eta) + v_k^-(\eta), \quad v_k^+(\eta) = v_k^-(-\eta).$$

(4.37)

where the amplitude of the module functions are undetermined. Therefore, the general solution can be written as

$$v(x) = \int \frac{d^3k}{(2\pi)^3} \left( v_k^+ e^{i\omega_k} + v_k^- e^{-i\omega_k} \right) e^{ik \cdot x} = \int \frac{d^3k}{(2\pi)^3} \left( \hat{v}_k^+ e^{-ikx} + \hat{v}_k^- e^{ikx} \right), \quad k := (\omega_k, \mathbf{k}).$$

(4.38)

### 4.2.2 Canonical Quantization

So far, we have derived a classical solution of the quadratic action for scalars. By promoting the Mukhanov-Sasaki field $v$ and its canonical momentum field $\pi$ to quantum fields, we need to impose the canonical quantization relation ($\hbar = 1$)

$$[\hat{v}(\eta, x), \hat{\pi}(\eta, y)] = i\delta^{\text{3D}}(x - y), \quad [\hat{v}(\eta, x), \hat{v}(\eta, y)] = [\hat{\pi}(\eta, x), \hat{\pi}(\eta, y)] = 0,$$

(4.39)

where we work in the Heisenberg picture for the time-dependent operators. Apparent from the notation, we want to define the creation and annihilation operators as

$$v_k^- := \hat{a}_kv_k^-, \quad (v_k^-)^\dagger = v_k^+ = \hat{a}_k^\dagger v_k^+, \quad (v_k^-)^* = v_k^+.$$  

(4.40)

\[\text{Here, "scalars" are used to refer to the scalar fluctuations, not to be confused with the scalar field.}\]
such that we derive
\[ \hat{v}(x) = \int \frac{d^3k}{(2\pi)^3} \left( \hat{a}^+_k v^+_k e^{-ikx} + \hat{a}_k v^-_k e^{ikx} \right) = \int \frac{d^3k}{(2\pi)^3} \left[ \hat{a}^+_k v^+_k(\eta) e^{-ik\cdot\eta} + \hat{a}_k v^-_k(\eta) e^{ik\cdot\eta} \right], \]  
where we defined
\[ v^+_k(\eta) := v^+_k e^{i\omega_k \eta}. \]
By substituting into the canonical quantization relation, we can derive that the ladder operators indeed satisfy the standard quantization relation at the equal time
\[ [\hat{a}_k, \hat{a}^+_k] = (2\pi)^3 \delta^{3D}(k - k'), \quad [\hat{a}_k, \hat{a}_k'] = [\hat{a}^+_k, \hat{a}^+_k] = 0, \]
if the normalization for the mode functions is properly chosen
\[ W[v^-_k, v^+_k] := v^-_k(\eta) v^{+*}_k(\eta) - v^{+*}_k(\eta) v^-_k(\eta) = i. \]  
With the properly normalized operators, we obtain the usual relations
\[ \hat{a}_k |0\rangle = 0, \quad (0|0) = 1, \quad |n_k\rangle = \frac{\sqrt{2E_k}}{n!} [(\hat{a}^+_k)^n] |0\rangle, \]
where \( \sqrt{2E} \) is put to make it Lorentz invariant. One can quantize the field, starting with the time-independent Harmonic oscillators, then applying the Heisenberg picture with the free-field Hamiltonian, as in Peskin & Schröder.

### 4.2.3 Vacuum Expectation Value
While we imposed the normalization condition for the mode functions in terms of their Wronskian, the physical vacuum is yet to be fully determined, due to the arbitrariness in the mode functions. Consider a different set of mode functions \( u^\pm_k \) that are related to the original mode functions as
\[ u^-_k(\eta) = \alpha_k v^-_k(\eta) + \beta_k v^+_k(\eta), \]
and construct the creation and annihilation operators \( \hat{b}^\pm_k \) with \( u^\pm_k \)
\[ u^-_k := \hat{b}^-_k u^-_k. \]
Using this relation, we can write the operator \( \hat{v} \) and its canonical momentum \( \hat{\pi} \) in terms of \( \hat{b}_k \) and \( \hat{b}^+_k \). These two sets of quantum operators are then related as by, so called, the Bogolyubov transformation:
\[ \hat{a}_k = \alpha_k \hat{b}_k + \beta_k \hat{b}^-_{-k}, \quad \hat{a}^+_k = \alpha_k \hat{b}^+_k + \beta_k \hat{b}^-_{-k}, \quad |\alpha_k|^2 - |\beta_k|^2 = 1, \]
where the normalization for the transformation coefficients is due to the Wronskian normalization. Note that the vacuum defined by one set of operators \( \hat{a}_k \) is not the vacuum with respect to the other set of operators \( \hat{b}_k \). To properly determine the physical vacuum, we need to fix the mode function completely.
In terms of the mode functions, the Hamiltonian in Minkowski spacetime is
\[ \hat{H} = \int d^3x \ \hat{H}, \quad \hat{H} = \frac{1}{2} \left[ \hat{\pi}^2 + (\nabla \hat{v})^2 \right], \quad m \to 0. \]
Using the expression for the mode function in Eq. (4.1), we derive the Hamiltonian
\[ \hat{H} = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left[ \left( (v^+_k)^2 + k^2 (v^-_k)^2 \right) \hat{a}^+_k \hat{a}_k \hat{a}^+_k \hat{a}_k \right. \]
\[ + \left. \left( (v^-_k)^2 + k^2 (v^+_k)^2 \right) \hat{a}_k \hat{a}^-_k \hat{a}_k \hat{a}^-_k \right], \]
acting on the vacuum \( |0\rangle \) as
\[ \hat{H} |0\rangle = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left[ \left( (v^+_k)^2 + k^2 (v^-_k)^2 \right) \hat{a}^+_k \hat{a}^+_k \hat{a}^-_k \hat{a}^-_k \right] |0\rangle. \]
The vacuum \(|0\rangle\) is an eigenstate of the Hamiltonian, and indeed the first round bracket vanishes. Given the normalization of the Wronskian, the physical mode function is then
\[
W[v_k, v_k'] = 2ik|v_k'|^2 = i, \quad v_k'(\eta) = \frac{1}{\sqrt{2k}} e^{-ik\eta}.
\]
Therefore, we derive the vacuum expectation values
\[
\langle 0|\hat{v}_k^* \hat{v}_k|0\rangle = (2\pi)^3 \delta^{3D}(k - k')P_v(k), \quad P_v(k) = |v_k'|^2 = \frac{1}{2k}.
\]

4.2.4 Scalar Fluctuations

Now we consider the time-dependent mass term in the equation of motion, and following the same procedure we pick the vacuum that corresponds to the solution in the Minkowski spacetime as the modes were deep inside the horizon in the far past
\[
v_k(\eta) = \frac{1}{\sqrt{2\omega_k(\eta)}} e^{-i\omega_k(\eta)\eta}, \quad \lim_{\eta \to -\infty} v_k(\eta) = \frac{1}{\sqrt{2k}} e^{-ik\eta},
\]
and this choice is called the Bunch-Davis vacuum. To the zero-th order in the slow-roll approximation \((\varepsilon = 0)\), the inflationary period is the de-Sitter spacetime, in which
\[
m^2(\eta) = \frac{a''}{a} = \frac{2}{\eta^2}, \quad \omega_k^2 = k^2 - \frac{2}{\eta^2},
\]
and we can derive the exact solution for the mode functions:
\[
v_k(\eta) = e^{-ik\eta} \sqrt{\frac{2}{k\eta}} \left(1 - \frac{i}{k\eta}\right).
\]
When the \(k\)-mode is stretched beyond the horizon, the amplitude of the mode function is
\[
\lim_{k\eta \to 0} v_k(\eta) = \frac{1}{i\sqrt{2}} \frac{1}{k^{3/2}\eta}, \quad \lim_{k\eta \to 0} k^3|v_k|^2 = \frac{1}{2\eta^2} = \frac{a^2H^2}{2},
\]
and the power spectra of the mode function and the comoving-gauge curvature are
\[
P_v \equiv |v_k|^2 = \frac{a^2H^2}{2k^3}, \quad \Delta_t^2 \equiv \frac{k^3}{2\pi^2}P_t = \frac{1}{2a^2\varepsilon} \Delta_s^2 = \frac{H^2}{8\pi^2\varepsilon}.
\]

4.2.5 Tensor Fluctuations: Gravity Waves

We can repeat the exercise for the scalar fluctuations to derive the tensor fluctuations. The quadratic action for tensor is
\[
S_{(2)} = \frac{M_{pl}^2}{8} \int d\eta \, d^4\mathbf{x} \, a^2 \left[(h_{ij}')^2 - (\nabla h_{ij})^2\right] = \sum_{s = \pm 2} \int d\eta \, d^3\mathbf{k} \, a^2 M_{pl}^2 \left[(h_{ik}')^2 - k^2(h_{ik})^2\right],
\]
where we again decomposed the tensor in terms of two helicity eigenstates
\[
h_{ij} := 2C_{ij}^{(t)} = 2h(\pm 2)Q_{ij}^{(\pm 2)}.
\]
From the action, the Mukhanov-Sasaki variable for tensor fluctuations is
\[
v_k^s := \frac{a}{2} M_{pl} h_k^s, \quad m = 0,
\]
and we can readily derive the tensor power spectrum
\[
P_v = \frac{(aH)^2}{2k^3}, \quad P_T := 2P_{h^t_k} = 2 \left(\frac{2}{aM_{pl}}\right)^2 P_v = \frac{4}{k^3} \frac{H^2}{M_{pl}^2}.
\]
The amplitude of the tensor power spectrum is the energy scale of the inflation in the early Universe, and its ratio to the scalar power spectrum is
\[
r := \frac{\Delta_t^2}{\Delta_s^2} = \frac{8}{M_{pl}^2} \frac{\dot{\phi}^2}{H^2} = 16\varepsilon,
\]
slow-roll suppressed.
4.3 Predictions of the Standard Inflationary Models

4.3.1 Consistency Relations

For the standard single field inflationary models with the slow-roll approximation, we summarize the predictions for scalar fluctuations

\[ P_\zeta = \left( \frac{2\pi^2}{k^3} \right) A_s, \quad A_s := \frac{H^2}{8\pi^2\varepsilon M_{pl}^2} = \frac{1}{24\pi^2 \varepsilon} \frac{V}{M_{pl}^4}, \]  

\[ n_s - 1 := \frac{d \ln k^3 P_\zeta}{d \ln k} = (-2\varepsilon - \varepsilon^2)(1 - \varepsilon)^{-1} \simeq 2\eta_V - 6\varepsilon_V, \]  

the predictions for tensor fluctuations

\[ P_T = \frac{4}{k^3 M_{pl}^2} = \left( \frac{2\pi^2}{k^3} \right) A_T, \quad A_T := \frac{2}{\pi^2} \frac{H^2}{M_{pl}^2} = \frac{2V}{3\pi^2 M_{pl}^4}, \quad n_t := \frac{d \ln k^3 P_T}{d \ln k} \simeq -2\varepsilon, \]  

and the consistency relations

\[ r := \frac{A_T}{A_s} = \frac{8\dot{\phi}_k^2}{H^2} = 16\varepsilon = -8n_t. \]  

By measuring the power spectrum amplitude and its slope for both scalar and tensor fluctuations, we can ensure that the fluctuations are indeed generated by a single field inflaton or rule out the standard inflationary models. There exist other predictions in the standard inflationary models (and of course, for the beyond the standard models) that can be used to test models, such as the primordial non-Gaussianity and so on.

4.3.2 Lyth Bound

Given the definition of the e-folds, we can further manipulate it by using the inflaton as a time clock:

\[ N(\phi_k) = \int_{\phi_k}^{\phi_{\text{end}}} d\phi \frac{H}{\dot{\phi}} = \int_{\phi_k}^{\phi_{\text{end}}} \frac{d\phi}{M_{pl}\sqrt{2\varepsilon}}, \quad r = 16\varepsilon = \frac{8}{M_{pl}^2} \left( \frac{d\phi}{dN} \right)^2, \]  

and this relation further implies that the excursion of the inflaton field is related to the tensor-to-scalar ratio as

\[ \frac{\Delta \phi_k}{M_{pl}} \simeq \int_{N_{\text{end}}}^{N_{\text{emb}}} dN \sqrt{\frac{r}{8}}, \]  

where \( \varepsilon(\phi_{\text{end}}) \equiv 1 \). To solve the horizon problem, the mode \( k \) should have expanded at least 40–60 in e-folds. So, this consistency relation (Lyth, 1997) implies that an inflationary field variation of the order of the Planck mass is needed to produce \( r > 0.01 \). From the theoretical point of view, this sets the upper bound on the amplitude of gravitational waves. Indeed, the standard inflationary model predictions are very small.

Note that the uncertainty in e-folds \( N \) is due to our ignorance in the reheating era: After the inflationary period ends, the inflaton field decays into other particles and reheats the Universe. This period is expected to be described by a matter-dominated era, as the inflaton oscillates around the minimum of the potential, effectively acting as a matter. However, we know very little about this period.

The current observational constraint is

\[ A_s \simeq 2.2 \times 10^{-9}, \quad n_s \simeq 0.96, \quad \varepsilon \simeq 0.01. \]  

indicating the energy scale of the inflation is

\[ A_T = 2V \left( \frac{1}{3\pi^2 M_{pl}^4} \right) = 16\varepsilon A_s, \quad H^2 = \frac{V}{3M_{pl}^2} = \varepsilon (2 \times 10^{14} \text{GeV})^2. \]
4.3.3 A Worked Example

Here we consider a very simple inflationary model with a power-law potential:

\[ V = \frac{1}{2} m^4 \phi^4, \]  

(4.72)

where the mass \( m \) and the slope \( \alpha \) are the free parameters of the model. It chaotically starts everywhere at any time in field configurations, and its predictions are then

\[ \varepsilon_V = \frac{\alpha^2}{2} \left( \frac{M_{\text{pl}}}{\phi} \right)^2, \quad \eta_V = \alpha (\alpha - 1) \left( \frac{M_{\text{pl}}}{\phi} \right)^2, \]  

(4.73)

\[ N \simeq \int \frac{d\phi}{M_{\text{pl}}^2 V'} = \frac{\phi^2 - \phi_{\text{end}}^2}{2M_{\text{pl}}^2 \alpha}, \quad r \simeq 16 \varepsilon_V, \quad n_s - 1 \simeq 2\eta_V - 6\varepsilon_V. \]  

(4.74)

Approximating \( \phi_{\text{end}} \simeq 0 \), we further derive

\[ N \simeq \frac{1}{2\alpha} \left( \frac{\phi}{M_{\text{pl}}} \right)^2, \quad \varepsilon_V \simeq \frac{\alpha}{4N}, \quad \eta_V = \frac{\alpha - 1}{2N}, \quad 1 - n_s \simeq \frac{\alpha + 2}{2N}, \quad r \simeq \frac{4\alpha}{N}. \]  

(4.75)

4.4 Adiabatic Modes and Isocurvature Modes

- **Adiabatic modes.**— Assuming a flat Universe, we can arrange Eq. (4.28) to show

\[ \dot{\Phi} = \Xi - \frac{H}{\rho + p} a^2 \left( \frac{c_s^2}{4\pi G} \varphi - \frac{2}{3} \Pi \right), \quad \Xi := \frac{\dot{\rho}}{3(\rho + p)} \equiv -\frac{He}{\rho + p}. \]  

(4.76)

Therefore, in the limit \( k \to 0 \), if \( \Xi = 0 \) vanishes, the comoving-gauge curvature perturbation is conserved, regardless of contents in the Universe. Indeed, \( \Xi = 0 \) if the pressure is a function of the density, and it holds true for the matter-dominated era, the radiation-dominated era, and for the single field inflation.\(^3\) This condition is called adiabatic, because individual components fluctuate at the same rate at a given point:

\[ \frac{\delta \rho_i}{\rho_i} = \frac{\delta \rho_{\text{tot}}}{\rho_{\text{tot}}} = \frac{\delta p_i}{p_i} = \frac{\delta p_{\text{tot}}}{p_{\text{tot}}} \equiv -\varphi \mathcal{I}, \quad \frac{\delta_a}{1 + w_a} = \frac{\delta_b}{1 + w_b} \text{ for } \forall a, b. \]  

(4.77)

Even for single-field inflationary scenarios, there should have existed many other matter fields, and some energy transfer to these fields are inevitable. However, these non-adiabatic perturbations decay fast as the inflation proceeds, and they become exponentially suppressed when these matter fields dominate the energy budget during the reheating era.

In the limit \( k \to 0 \), we can indeed derive the adiabatic condition

\[ \mathcal{I} := \frac{1}{a} \int_t^\infty dt \, a(t), \quad v_\chi \equiv -\frac{1}{a} \mathcal{I} \varphi_v. \]  

(4.78)

- **Isocurvature mode.**— The evolution of isocurvature perturbations depends not only on inflationary dynamics, but also on post-inflationary evolution. For example, if all particles thermalize after inflation, all isocurvature perturbations become adiabatic perturbations eventually. The isocurvature perturbations and the entropy perturbations are interchangeably used, because they do represent the perturbations between species and it does conserve the curvature. In practice, the entropy perturbations are parametrized by two free parameters at some pivot scale \( k_0 \) (0.002/Mpc in WMAP), i.e., ratio \( \alpha \) of the isocurvature to the adiabatic perturbations and their correlation \( \beta \)

\[ \frac{P_S}{P_\zeta} := \frac{\alpha}{1 - \alpha}, \quad \beta := \frac{P_{S\zeta}}{\sqrt{P_SP_\zeta}}, \]  

(4.79)

where the relative entropy perturbation (or specific entropy) is defined as

\[ S_{XY} = \delta \left( \frac{n_X}{n_Y} \right) / \left( \frac{n_X}{n_Y} \right) = \delta n_X / n_X - \delta n_Y / n_Y = \frac{\delta_X}{1 + w_X} - \frac{\delta_Y}{1 + w_Y}. \]  

(4.80)

\(^3\)It vanished only in the limit \( k = 0 \) for single field models.
By defining the gauge-invariant curvature perturbation in the uniform-density gauge
\[ \varphi_\delta = \varphi - \frac{H \delta \rho}{\dot{\rho}} = \varphi + \frac{\delta}{3(1 + w)} , \]  
(4.81)
we can readily show that the entropy perturbation is gauge invariant
\[ S_{XY} = 3 (\varphi^X_\delta - \varphi^Y_\delta) . \]  
(4.82)

In literature, it is often the case that the species $Y$ is reserved for photons.

- PNG, multi-field, delta-N formalism, curvaton
5 Boltzmann Equations and CMB Temperature Anisotropies

5.1 Boltzmann Equation

5.1.1 Collisionless Boltzmann Equation

The Liouville theorem in GR states that the phase-space volume is conserved along the path parametrized by \( \lambda \) with momentum \( p^\mu \):

\[
0 = \Delta(dN) = \left( \frac{\partial f}{\partial x^\mu} \Delta x^\mu + \frac{\partial f}{\partial p^\mu} \Delta p^\mu \right) dV_p = \left( p^\mu \frac{\partial f}{\partial x^\mu} - \Gamma^\mu_{\rho\sigma} p^\rho p^\sigma \frac{\partial f}{\partial p^\mu} \right) \Delta \lambda \ dV_p . \tag{5.1}
\]

This translates into the relativistic collisionless Boltzmann equation:

\[
0 = p^\mu \frac{\partial f}{\partial x^\mu} - \Gamma^\mu_{\rho\sigma} p^\rho p^\sigma \frac{\partial f}{\partial p^\mu} , \quad \text{or} \quad 0 = p^\mu \frac{\partial f}{\partial x^\mu} + \Gamma^\mu_{\rho\sigma} p^\rho p^\sigma \frac{\partial f}{\partial p^\mu} , \tag{5.2}
\]

where we used the geodesic equation

\[
0 = \frac{d}{d\lambda} p^\mu - \Gamma^\mu_{\rho\sigma} p^\rho p^\sigma . \tag{5.3}
\]

Despite the presence of the Christoffel symbol, the equation is indeed invariant under diffeomorphisms. We further need to impose the on-shell condition in the collisionless Boltzmann equation.

5.1.2 Linear-Order Geodesic Equation

To solve the Boltzmann equation (5.2), we need to know how the physical momentum \( p^i \) in FRW coordinates is related to the physical quantities measured by an observer with four velocity \( u^i \).

\[
[e_i]^\mu = u^\mu = \frac{1}{a} (1 - A, U^\alpha) , \quad [e_i]^\mu = \frac{1}{a} \left[ \delta_i^\beta \left( U_\beta - B_\beta \right) , \delta_i^\beta - \delta_i^\alpha \left( \varphi^\alpha + G_{\alpha\beta} + C_{\alpha\beta} \right) - \epsilon_i^{\alpha\beta} \Omega^j \right] , \tag{5.4}
\]

where \( \Omega^i \) is the rotation of tetrad vectors against FRW coordinates. The physical momentum is written in capital letters as

\[
P^\alpha = (E, P^i) , \quad E = -p^\mu u_\mu , \quad P^i = p^\mu e_i^\mu , \quad E^2 = m^2 + P^2 . \tag{5.5}
\]

Using the tetrad expression, the physical momentum \( p^\mu = P^\alpha e_\alpha^\mu \) in FRW coordinates is then obtained as

\[
p^\eta = \frac{(1 - A) E + (U_j - B_j) P^j}{a} , \quad p^\alpha = \frac{1}{a} \left[ P^\alpha + E U^\alpha - P^\beta \left( \varphi^\alpha_{\beta\gamma} + G_{\alpha\beta} + C_{\alpha\beta} \right) - \epsilon_{\alpha\beta} P^\beta \Omega^j \right] , \tag{5.6}
\]

and the covariant momentum is

\[
p_\eta = -a (1 + A) E - a U_j P^j , \quad p_\alpha = a \left( U_\alpha - B_\alpha \right) E + a \left[ P_\alpha \left( 1 + \varphi \right) + (G_{\beta\alpha} + C_{\alpha\beta}) P^\beta - \epsilon_{\alpha\beta} P^\beta \Omega^j \right] . \tag{5.7}
\]

In the background, the physical momentum is redshifted as \( P_\alpha \propto 1/a \) for both massless and massive particles.\(^1\) So, it is convenient to define the “comoving momentum \( q \)” and “comoving energy \( \varepsilon \)” as

\[
q := aP , \quad \varepsilon := aE = \sqrt{q^2 + a^2 m^2} , \quad q^i := q n^i . \tag{5.9}
\]

\(^1\)The geodesic equation yields

\[
0 = p^\mu p^\nu + \Gamma^\mu_{\rho\sigma} p^\rho p^\sigma \rightarrow 0 = p^\mu p^\nu + \mathcal{H} p^\mu p^\sigma , \quad 0 = p^\mu p^\nu + 2\mathcal{H} p^\mu p^\sigma . \tag{5.8}
\]

The last equation says the spatial momentum \( p^\nu \propto 1/a^2 \) in the background, i.e., the physical momentum \( P_\alpha \propto 1/a \) for both massless and massive particles in the background. In the presence of perturbations, these relations change.
In the background, the comoving momentum and energy are constant, while the momentum in FRW coordinates redshifts as
\[ \tilde{p}^\eta = \frac{E}{a} = \frac{\varepsilon}{a^2}, \quad \tilde{p}^\alpha = \frac{1}{a} P^\alpha = \frac{1}{a^2} q n^\alpha, \quad n^\alpha = n^i \delta_i^\alpha. \] (5.10)

In terms of the comoving quantities, the momentum in FRW coordinates is now
\[ p^\eta = \left(1 - A\right) \varepsilon + (U_j - B_j) q^j, \quad p^\alpha = \frac{1}{a^2} \left[q^\alpha + \varepsilon U^\alpha - q^j \left(\varphi \delta^\beta_j + G^\alpha_{\beta j} + C^\alpha_{\beta j}\right) - \epsilon_i j q^i \Omega'\right]. \] (5.11)

To compute the change in the comoving momentum as the particle propagates, we need to solve the geodesic equation and obtain
\[ \frac{dq^\alpha}{d\eta} = -\varepsilon A^\alpha - (U^\alpha - B^\alpha) \varepsilon - q^\beta \left(U_{\alpha,\beta} - B_{\alpha,\beta} + B_{\beta,\alpha}\right) - \frac{a^2 H m^2}{\varepsilon} (U_{\alpha} - B_{\alpha}) - \varphi^\prime q_\alpha - q^\beta \left(G^\prime_{\beta,\alpha} + C_{\alpha,\beta}\right) - \frac{q^\beta q^\gamma}{\varepsilon} \left(\varphi \delta^\beta_{\gamma\delta} + C_{\alpha\beta\gamma} - C_{\beta\gamma\alpha}\right), \] (5.12)

where we use the background geodesic equation (valid for massive & massless)
\[ \frac{d}{d\eta} = \frac{\partial}{\partial \eta} + \frac{q^\beta}{\varepsilon} \frac{\partial}{\partial x^\beta}, \quad \varepsilon' = \frac{a^2 H m^2}{\varepsilon}. \] (5.13)

At the linear order, the propagation direction is simply the straight path, and only the comoving momentum changes as
\[ \frac{d \ln q}{d\eta} = -\frac{\varepsilon}{q} \left[A_{\parallel} + (U - B)_{\parallel}\right] - \varphi' - U_{\alpha,\beta} n^\alpha n^\beta - \frac{a^2 H m^2}{q\varepsilon} (U - B)_{\parallel} - (G'_{\beta,\alpha} + C_{\alpha,\beta}) n^\alpha n^\beta. \] (5.14)

Indeed, the comoving momentum is constant in the background.

### 5.1.3 Linear-Order Boltzmann Equation

The Boltzmann equation is further simplified, when we switch the variables \((\eta, x^\alpha, p^\mu)\) to \((\eta, x^\alpha, q^i)\), where the on-shell condition removes one component of the physical momentum:
\[ 0 = \frac{df}{d\Lambda} \Delta \Lambda = \left(\frac{\partial f}{\partial x^\mu} \Delta x^\mu + \frac{\partial f}{\partial q^i} \Delta q^i\right) \Delta \Lambda = \left(p^\mu \frac{\partial f}{\partial x^\mu} + \frac{dq^i}{d\ln q} \frac{\partial f}{\partial q^i}\right) \Delta \Lambda, \] (5.15)

where the partial derivatives fix \((x^\mu, q^i)\), instead of \((x^\mu, p^\mu)\) in Eq. (5.2). Splitting the distribution function \(F\), we derive the Boltzmann equation in the background:
\[ F := \tilde{f} + f, \quad \tilde{f} = f', \quad \tilde{f} = \tilde{f}(q), \] (5.16)

i.e., the phase-space distribution is constant in time and space, but a function of the comoving momentum only. The perturbation equation can be derived as
\[ 0 = \left(\tilde{p}^0 f' + \tilde{p}^\alpha f_{,\alpha} + \frac{dq^i}{d\Lambda} n_i \frac{df}{dq}\right) \Delta \Lambda + \mathcal{O}(2) = \tilde{p}^0 \left(f' + \frac{q n^\alpha}{\varepsilon} \frac{\partial f}{\partial x^\alpha} + \frac{df}{d\ln q} \frac{a^2 n_i dq^i}{q d\Lambda}\right) \Delta \Lambda, \] (5.17)

where the last term is
\[ \frac{d}{d\eta} = \frac{1}{p^0} \frac{d}{d\Lambda}, \quad \frac{dx^\alpha}{dq} = \frac{1}{p^0} \frac{dx^\alpha}{d\Lambda} = \frac{p^\alpha}{p^0}, \quad \frac{df}{d\ln q} \frac{a^2 n_i dq^i}{q d\Lambda} = \frac{df}{d\ln q} \frac{n_i dq^i}{q d\eta}. \] (5.18)

Therefore, the collisionless Boltzmann equation is at the linear order in perturbations
\[ 0 = f' + \frac{q n^\alpha}{\varepsilon} \frac{\partial f}{\partial x^\alpha} - \frac{df}{d\ln q} \left(\frac{\varepsilon}{q} A_{\parallel} + \left(\frac{\varepsilon}{q} (U - B)_{\parallel}\right)\right)' + \varphi' + n^\alpha n^\beta (U_{\alpha,\beta} + G'_{\beta,\alpha} + C'_{\alpha,\beta}), \] (5.19)
where we used $X_{||} = X_\alpha n^\alpha$. In the final equation, the orientation $\Omega^i$ of the local rest-frame disappears, the vector part becomes symmetric.

Given the transformation properties under diffeomorphisms and Lorentz transformations

$$\delta T = -\mathcal{H} T \frac{df}{d\ln q}, \quad \delta g = \frac{\epsilon}{q} n^i \theta^{0i} \frac{df}{d\ln q},$$

we can construct a fully gauge-invariant combination:

$$f_g = f - \left( H \chi + \frac{\epsilon}{q} V_{||} \right) \frac{df}{d\ln q}, \quad \frac{d}{d\eta} f_g = f'_g + \frac{q n^\alpha}{\epsilon} \frac{\partial f_g}{\partial x^\alpha}, \quad \delta g V^i = \theta^{0i}.$$

By substituting into the Boltzmann equation, we derive

$$0 = \frac{d}{d\eta} f_g - \frac{df}{d\ln q} \left[ \frac{\epsilon}{q} \alpha_{X,||} + \varphi'_{X} + n^\alpha n^\beta \left( \Psi_{\alpha,\beta} + C'_{\alpha,\beta} \right) \right],$$

where we used the $a^2 m^2 = \epsilon^2 - q^2$ and

$$A_{\alpha} + U'_{\alpha} - B'_{\alpha} = (\alpha_X + H \chi)_{\alpha} + V'_{\alpha}, \quad U_{\alpha} - B_{\alpha} = V_{\alpha} - \frac{1}{a} \chi_{\alpha}, \quad U_{\alpha} + G'_{\alpha} = V_{\alpha} + \Psi_{\alpha}.$$

To make it apparent which part in the source (square bracket) gets pulled out to combine into the total derivative $d\eta$ to make $f$ gauge-invariant, we re-write

$$0 = f' + \frac{q n^\alpha}{\epsilon} \frac{\partial f}{\partial x^\alpha} - \frac{df}{d\ln q} \left[ \frac{\epsilon}{q} \alpha_{X,||} + \varphi'_{X} + n^\alpha n^\beta \left( \Psi_{\alpha,\beta} + C'_{\alpha,\beta} \right) \right].$$

The collision term should be set equal to the above Liouville (collisionless Boltzmann) equation, but with $\tilde{p}^\alpha = \epsilon/a^2$ in the denominator.

## 5.2 Multipole Expansion of the Boltzmann Equation

### 5.2.1 Decomposition Convention

Now we will decompose the Boltzmann equation, but care must be taken in terms of what variables are decomposed. Schematically, we will perform the decomposition of the perturbations in the Boltzmann equation as

$$P(x^\mu, q^i) = \int \frac{d^3 k}{(2\pi)^3} e^{i k \cdot x} P(k, q^i, \eta) = \int \frac{d^3 k}{(2\pi)^3} \sum_{l,m} (-i)^l \sqrt{\frac{4\pi}{2l + 1}} P_{lm}(q, k, \eta) Y_{lm}(\hat{n}) e^{i k \cdot x},$$

where the angular decomposition is (we suppress irrelevant arguments for clarity such as $\eta$ and $k$)

$$P(q^i) := \sum_{l,m} (-i)^l \sqrt{\frac{4\pi}{2l + 1}} P_{lm}(q) Y_{lm}(\hat{n}) , \quad P_{lm}(q) = i^l \sqrt{\frac{2l + 1}{4\pi}} \int d^2 \eta Y_{lm}^*(\hat{n}) P(q^i).$$

Naturally, $P_{lm}$ are helicity eigenstates, such that under a rotation $\phi \rightarrow \phi - \Phi$ in a coordinate $(k||z)$ they transform as

$$\tilde{P}_{lm} = P_{lm} e^{im\Phi}.$$ 

Since we integrate over $\hat{n}$ and $\hat{k}$, we can simply set $k//z$.

In literature, there exists a different convention (up to $2l + 1$ factor) for decomposition in terms of the Legendre polynomial, but it is valid only for a scalar mode ($m = 0$);

$$P(\hat{k} \cdot \hat{n}) := \sum_l (-i)^l P_l L_l(\hat{n} \cdot \hat{k}) = \sum_l (-i)^l P_l \sum_m \frac{4\pi}{2l + 1} Y_{lm}(\hat{n}) Y_{lm}^*(\hat{k}),$$

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and for a Fourier mode $k \parallel z$ we derive the correspondence to our decomposition convention:

$$Y_{lm}(z) = \sqrt{\frac{2l+1}{4\pi}} \delta_{m0}, \quad P(k \cdot \hat{n}) = \sum_{l} (-i)^l P_l \delta_{m0} \sqrt{\frac{4\pi}{2l+1}} Y_{lm}(\hat{n}) \, . \quad \therefore \quad P_l \rightarrow P_{l0} \ . \quad (5.29)$$

So the decomposition with the Legendre polynomial in Eq. (5.28) is valid only for the scalar modes. Here, we will also focus on the scalar modes to simplify the calculations in the following.

hereafter scalar modes only! \hspace{1cm} (5.30)

### 5.2.2 Useful Relations

Since we will work with the helicity eigenstates, we will use the following decomposition convention:

$$P(x^{\mu}) \rightarrow P_{k(0)} Q, \quad P_{\alpha(\nu)}(x^{\mu}) \rightarrow P_{k(\pm 1)} Q_{\alpha(\nu)}^{(\pm 1)}, \quad P_{\alpha(\beta)}(x^{\mu}) \rightarrow P_{k(\pm 2)} Q_{\alpha(\beta)}^{(\pm 2)} \ . \quad (5.31)$$

With the scalar quantity $Q = Q^{(0)} = e^{ikz}$, we construct the vector and the tensor harmonics

$$Q^{(0)}_{\alpha} = -\frac{1}{k} \nabla_{\alpha} Q = (0, 0, -i)Q, \quad Q^{(0)}_{\alpha \beta} = \frac{1}{k^2} Q^{(0)}_{\alpha} + \frac{1}{3} \bar{g}_{\alpha \beta} Q = \left( \begin{array}{ccc} 1/3 & 0 & 0 \\
0 & 1/3 & 0 \\
0 & 0 & -2/3 \end{array} \right) Q \ . \quad (5.32)$$

So, we have the usual vectors

$$v^\alpha = v^{(0)} Q^{(0)}_{\alpha} = (0, 0, -i v^0), \quad U^{(0)} = k U, \quad B^{(0)} = -k \beta, \quad (U-B)^{(0)} = k \left( v^\alpha + \frac{1}{\alpha} \chi \right), \quad V^{(0)} = k v^\chi \ . \quad (5.33)$$

The spatial part of the metric tensor is

$$\varphi \bar{g}_{\alpha \beta} + \gamma_{\alpha \beta} = \left( \varphi - \frac{k^2}{3} \right) \bar{g}_{\alpha \beta} Q^{(0)} + k^2 \gamma Q^{(0)}_{\alpha \beta}, \quad H_L = \varphi - \frac{1}{3} k^2 \gamma, \quad H_T^{(0)} = k^2 \gamma \ . \quad (5.34)$$

For the angular decomposition, we will use the following relations

$$n^i = \sqrt{\frac{4\pi}{3}} \left( \frac{Y_{l,-1} - Y_{l,1}}{\sqrt{2}}, \frac{Y_{l,-1} + Y_{l,1}}{\sqrt{2}}, Y_{10} \right), \quad \mu = \cos \theta = \sqrt{\frac{4\pi}{3}} Y_{10} \ , \quad (5.35)$$

$$n^\alpha n^\beta = \sqrt{\frac{4\pi}{3}} Y_{00} \delta_{\alpha \beta} + \sqrt{\frac{2\pi}{15}} \left( \begin{array}{ccc}
Y_{22} + Y_{2,-2} - \sqrt{2} Y_{20} & i(Y_{2,-2} - Y_{22}) & Y_{2,-1} - Y_{21} \\
i(Y_{2,-2} - Y_{22}) & -Y_{22} - Y_{2,-2} - \sqrt{2} Y_{20} & i(Y_{21} + Y_{2,-1}) \\
Y_{2,-1} - Y_{21} & i(Y_{21} + Y_{2,-1}) & \sqrt{8/3} Y_{20} \end{array} \right) \ , \quad (5.36)$$

and the integral identity

$$\int d^2 n n^i Y_{lm}^*(\hat{n}) = \sqrt{\frac{4\pi}{3}} \delta_{l1} \left( \frac{\delta_{m,-1} - \delta_{m1}}{\sqrt{2}}, i \frac{\delta_{m,-1} + \delta_{m1}}{\sqrt{2}}, \delta_{m0} \right) \ , \quad (5.37)$$

$$\int d^2 n Y_{lm}^* Y_{lm'} \mu = \sqrt{\frac{l^2 - m^2}{(2l-1)(2l+1)}} \delta_{l,l'-1} \delta_{mm'} \sqrt{\frac{(l+1)^2 - m^2}{(2l+1)(2l+3)}} \delta_{l,l'+1} \delta_{mm'} \ . \quad (5.38)$$

### 5.2.3 Multipole Expansion of the Boltzmann Equation

The multipole decomposition of the Boltzmann equation becomes

$$0 = \int_{l0} \frac{kq}{\varepsilon} \left( - \frac{l}{2l-1} f_{l-1,0} + \frac{l+1}{2l+3} f_{l+1,0} \right) - \frac{d}{d \ln q} \left[ \left( \varphi - \frac{k^2}{3} \gamma \right) \delta_{l0} + \frac{1}{3} k^2 U \delta_{l0} - \frac{\varepsilon kA}{q} \delta_{l1} \right. \left. + \left( \frac{\varepsilon}{q} (U^{(0)} - B^{(0)}) \right) \delta_{l1} + \frac{2}{3} k^2 (\gamma' - U) \delta_{l2} \right] \ . \quad (5.39)$$
where the $m$-dependence was omitted but it can be read off from the perturbation helicity. Given the transformation properties of the distribution function under diffeomorphisms Lorentz transformations, the multipole coefficients transform as
\[
\begin{align*}
\delta f_{0m} &= \frac{i}{2} \sqrt{\frac{2l+1}{4\pi}} \int d^2n Y^* (\hat{n}) \left[ \delta \xi f \right] = -HT \delta_{10} \frac{df}{d\ln q}, \\
\delta f_{lm} &= \frac{i}{2} \sqrt{\frac{2l+1}{4\pi}} \int d^2n Y^* (\hat{n}) \left[ \delta \phi f \right] = i \frac{df}{d\ln q} \delta_{10} \Theta^{\mu} \left( \frac{\delta_m - \delta_{m1}}{\sqrt{2}}, i \frac{\delta_{m1} + \delta_{m1}}{\sqrt{2}}, \delta_{m0} \right),
\end{align*}
\]
(5.40)
\( \delta \phi f \)

So, the monopole $f_{00}$ changes only under diffeomorphisms and the dipole $f_{1m}$ changes only under Lorentz transformations, while the higher multipoles are fully gauge-invariant. We construct the fully gauge-invariant variables as
\[
\begin{align*}
f_{00}^{\xi} &:= f_{00} - H\chi \frac{df}{d\ln q}, \\
f_{1m}^{\xi} &:= f_{1m} - \frac{\varepsilon V^{(m)}}{q} \frac{df}{d\ln q}, \\
f_{lm}^{\xi} &:= \frac{i}{2} \sqrt{\frac{2l+1}{4\pi}} \int d^2n Y^* (\hat{n}) f_{\xi} = f_{lm} \quad \text{for} \quad l \geq 2.
\end{align*}
\]
(5.42)
\( f_{lm}^{\xi} \)

The Boltzmann equation is in terms of gauge-invariant expressions
\[
\begin{align*}
0 &= f_{l0}' + \frac{kq}{\varepsilon} \left( -\frac{l}{2l-1} f_{l-1,0} + \frac{l+1}{2l+3} f_{l+1,0} \right) - \frac{df}{d\ln q} \left[ \varphi' f_{00} - \frac{\varepsilon}{q} k\varphi \delta_{l1} \right] \\
&\quad + \left( H\chi \delta_{l1} - \frac{2}{\varepsilon} kH\chi \delta_{l1} \right) + \left( \frac{\varepsilon}{q} k\varepsilon \right)' \delta_{l1} - \frac{2}{3} k^2 v_\chi \delta_{l2} + \frac{1}{3} k^2 v_\chi \delta_{l0}.
\end{align*}
\]
(5.44)

Though general, it is convenient to re-write it explicitly for the monopole and the dipole moments
\[
\begin{align*}
0 &= f_{00}' + \frac{kq}{\varepsilon} \left( -\frac{l}{2l-1} f_{l-1,0} + \frac{l+1}{2l+3} f_{l+1,0} \right) - \frac{df}{d\ln q} \left[ \varphi' f_{00} - \frac{\varepsilon}{q} k\varphi \delta_{l1} \right] \\
&\quad + \left( H\chi \delta_{l1} - \frac{2}{\varepsilon} kH\chi \delta_{l1} \right) + \left( \frac{\varepsilon}{q} k\varepsilon \right)' \delta_{l1} - \frac{2}{3} k^2 v_\chi \delta_{l2} + \frac{1}{3} k^2 v_\chi \delta_{l0}.
\end{align*}
\]
(5.44)
\( f_{00}' \)

The monopole and the dipole are affected by the scalar (and the vector) perturbations. The quadrupole moment is independent of scalar perturbations, but affected by the vector and the tensor perturbations. The higher-order multipoles for $l > 2$ are automatically gauge-invariant
\[
0 = f_{20}' + \frac{kq}{\varepsilon} \left( -\frac{2}{3} f_{l0} + \frac{3}{3} f_{30} \right) - \frac{df}{d\ln q} \left[ -\frac{2}{3} k^2 v_\chi \right] = f_{20}' + \frac{kq}{\varepsilon} \left( -\frac{2}{3} f_{l0} + \frac{3}{3} f_{30} \right)
\]
(5.46)
\( f_{20}' \)

and they are not source by any metric perturbations in the absence of collision (e.g., neutrino distribution), but they are not damped either.

### 5.3 Energy Momentum Tensor

#### 5.3.1 Energy Momentum Tensor from the Distribution Function

Using the distribution function $F$ in the rest-frame of the observer, we construct the energy momentum tensor in the rest-frame (the subscript $F$ is omitted)
\[
T^{ab} = \int \frac{d^3p}{E} P^a P^b F, \\
T_{ab} \equiv \begin{pmatrix}
\rho & -q_x & -q_y & -q_z \\
-q_x & p + \pi_{xx} & \pi_{xy} & \pi_{xz} \\
-q_y & \pi_{yx} & p + \pi_{yy} & \pi_{yz} \\
-q_z & \pi_{zx} & \pi_{zy} & p + \pi_{zz}
\end{pmatrix}, \quad \text{Tr} \pi = 0.
\]
(5.48)
\( T^{ab} \)

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where the fluid quantities measured by the observer are

\[
\rho = g \int d^3 P \mathcal{E}, \quad p = \frac{g}{3} \int d^3 P \frac{P^2}{E} F, \quad q^i = g \int d^3 P \frac{P^i}{E} F, \quad \pi^{ij} = g \int d^3 P \frac{P^i}{E} \left( n^j \delta_{ij} - \frac{1}{3} \delta_{ij} \right) F,
\]

(5.49)

and the perturbed quantities are

\[
\delta \rho = \frac{g}{a^4} \int d^3 q \, \varepsilon f = \frac{4 \pi g}{a^4} \int_0^\infty dq \, q^2 \varepsilon f_{00}, \quad (5.50)
\]

\[
\delta p = \text{Tr} \delta T^{ij} = \frac{g}{3a^4} \int d^3 q \, \frac{q^2}{\varepsilon} f = \frac{4 \pi g}{3a^4} \int_0^\infty dq \, \frac{q^4}{\varepsilon} f_{00}, \quad (5.51)
\]

Being scalars, the density \(\rho\) and the pressure \(p\) are identical in both FRW or the rest-frame. For the spatial flux and the anisotropic pressure, we derive

\[
q^i = \frac{g}{a^4} \int d^3 q \, q^i f = \frac{4 \pi g}{3a^4} \int dq \, q^3 \left( \alpha^i_{11} - \frac{1}{2} \alpha_{11} \right), \quad (5.52)
\]

\[
\pi^{ij} = \frac{g}{a^4} \int d^3 q \, \frac{q^2}{\varepsilon} \left( n^i n^j - \frac{1}{3} \delta_{ij} \right) f := \Pi^{(m)} Q^{(m)}_{ij}, \quad (5.53)
\]

where the helicity modes are

\[
Q^{(0)} = \frac{4 \pi g}{3a^4} \int dq \, q^3 f_{10}, \quad \Pi^{(0)} = \frac{4 \pi g}{5a^4} \int_0^\infty dq \, \frac{q^4}{\varepsilon} f_{20}. \quad (5.54)
\]

However, their relation to those in FRW coordinates are indeed

\[
q_i = q_\mu \epsilon_i^\mu = Q_\alpha \delta^\alpha_i + \mathcal{O}(2), \quad \pi_{ij} = \pi_{\mu\nu} \epsilon_i^{\mu} \epsilon_j^{\nu} = \Pi_{\alpha \beta} \delta^\alpha_i \delta^\beta_j + \mathcal{O}(2), \quad (5.55)
\]

where we used

\[
q_\alpha := a Q_\alpha, \quad \pi_{\alpha \beta} := a^2 \Pi_{\alpha \beta}. \quad (5.56)
\]

Finally, the spatial energy flux measured by the observer is the relative velocity

\[
T^0_i = q_i = (\bar{\rho} + \bar{p}) F (U_i^F - U_i^{\text{obs}}), \quad (5.57)
\]

and it can be expressed in a FRW coordinate

\[
T^\eta_{\alpha} = (\bar{\rho} + \bar{p}) F \left( U_\alpha^{\text{obs}} - B_\alpha \right) + Q_\alpha + \mathcal{O}(2) = (\bar{\rho} + \bar{p}) F \left( U_\alpha^F - B_\alpha \right) \equiv (\bar{\rho} + \bar{p}) v_\alpha, \quad (5.58)
\]

corresponding to the usual definition of the velocity \(v_\alpha\). For multiple components, we can sum over individual energy-momentum tensors, but with the same observer

\[
T^0_i = \sum_F (\bar{\rho} + \bar{p}) F \left( U_i^F - U_i^{\text{obs}} \right), \quad (5.59)
\]

\[
T^\eta_\alpha = \sum_F (\bar{\rho} + \bar{p}) F \left( U_\alpha^F - B_\alpha \right) \equiv \sum_F (\bar{\rho} + \bar{p}) F v^F_\alpha \equiv (\bar{\rho} + \bar{p}) T v^T_\alpha, \quad (5.60)
\]

where we defined

\[
v^F_\alpha := U_\alpha^F - B_\alpha, \quad v^T_\alpha = \sum_F \frac{(\bar{\rho} + \bar{p}) F}{(\bar{\rho} + \bar{p}) T} v^F_\alpha. \quad (5.61)
\]

Given the decomposition of the distribution function in the rest-frame, we derive, for example for photons

\[
(\bar{\rho} + \bar{p})_\gamma v^{(m)}_\gamma = (\bar{\rho} + \bar{p})_\gamma v^{(m)}_\gamma + \frac{4 \pi g}{3a^4} \int dq \, q^3 f_{1m}, \quad v^{\text{obs}}_\gamma \equiv U^{\text{obs}}_\gamma - B_\gamma. \quad (5.62)
\]
5.3.2 Notation Convention in Literature

The decomposition for the anisotropic pressure in FRW is
\[
\pi_{\alpha\beta} := a^2 \Pi_{\alpha\beta} := \left( \Pi_{\alpha\beta} - \frac{1}{3} g_{\alpha\beta} \Delta \Pi \right) + a \Pi_{(\alpha\beta)} + a^2 \Pi_{\alpha\beta}^{(t)},
\]
(5.63)
\[
\Pi_{\alpha\beta} = \frac{k^2}{{a^2}} Q_{\alpha\beta}^{(0)} \Pi - \frac{1}{k} \Pi^{(\pm 1)} Q_{(\alpha\beta)}^{(\pm 1)} + \Pi^{(\pm 2)} Q_{\alpha\beta}^{(\pm 2)},
\]
(5.64)
hence we derive
\[
\pi_{ij} = \Pi^{(m)} Q_{ij}^{(m)}, \quad \Pi^{(0)} = \frac{k^2}{a^2} \Pi.
\]
(5.65)

In literature, the anisotropic pressure is often defined with \( \sigma \):
\[
(\bar{\rho} + \bar{p}) \sigma_{MB} \equiv - \left( \bar{k} \bar{k} \bar{\beta} - \frac{1}{3} \bar{\beta} \bar{\beta} \right) \pi_{\alpha\beta} = \frac{Q_{\alpha\beta}^{(0)}}{Q} \pi_{\alpha\beta} = \frac{2 k^2}{3 a^2} \Pi Q_{\alpha\beta}^{(0)}, \quad \pi_{\alpha\beta} = \Pi_{\alpha\beta} + \mathcal{O}(2),
\]
(5.66)
\[
\Pi_{jB} \equiv -(\bar{\rho} + \bar{p}) \left( \bar{k} \bar{k} \bar{j} - \frac{1}{3} \bar{\beta} \bar{j} \right) \sigma_{B} = (\bar{\rho} + \bar{p}) \sigma_{B} Q_{ij}^{(0)} \rightarrow \pi_{ij} = \Pi_{\alpha\beta} + \mathcal{O}(2),
\]
(5.67)
and hence we have the correspondence:
\[
\sigma_{MB} = \frac{2}{3} \frac{k^2}{a^2} \Pi \bar{\rho} + \bar{p} \rightarrow \frac{2}{5} \Theta_{10}, \quad \sigma_{B} = \frac{k^2}{a^2} \Pi \bar{\rho} + \bar{p} = \frac{3}{2} \sigma_{MB},
\]
(5.68)
where MB and B stand for Ma & Bertschinger and Baumman, and we used
\[
Q_{\alpha\beta}^{(0)} = \frac{1}{k^2} Q_{\alpha\beta} + \frac{1}{3} \bar{\beta} \bar{\beta} Q = - \left( \bar{k} \bar{k} \bar{\beta} - \frac{1}{3} \bar{\beta} \bar{\beta} \right) Q, \quad Q_{ij}^{(0)} Q^{ij(0)} = \frac{2}{3} Q^2.
\]
(5.69)

In MB, they further define \( \theta \):
\[
(\rho + p) \theta_{MB} := i k^3 T^0_j = - i k^3 (\rho + p) v_j = (\rho + p) k^2 v \rightarrow \theta_{MB} = k^2 v.
\]
(5.70)
Furthermore, their convention for multipole coefficients is also different:
\[
F_{MB} = \sum_{l} (-i)^l (2l + 1) F_l L_l \rightarrow 4 \Theta, \quad F_l \rightarrow \frac{4}{2l + 1} \Theta_{10}.
\]
(5.71)

5.3.3 Massless Particles

Here we consider photons and neutrinos. Though neutrinos are massive, massless neutrinos are in most cases a good approximation, with which equations are greatly simplified. We define the temperature anisotropies
\[
\rho = a T^4 = a T^4 \left( 1 + 4 \frac{\delta T}{T} \right) + \mathcal{O}(2), \quad \Theta(\hat{n}) := \frac{\delta T}{T} = \frac{1}{4} \frac{\delta \rho}{\bar{\rho}},
\]
(5.72)
and its multipole decomposition
\[
\Theta_{10} = \frac{\int dq \, q^3 f_{10} \left( \frac{\pi g}{a^4 \bar{\rho}} \int_{0}^{\infty} dq \, q^3 f_{10} \right) - \varepsilon = q.
\]
(5.73)
Note that the temperature anisotropies \( \Theta \) are related to the distribution function as
\[
F = \left[ \exp \left( \frac{q}{a T(\eta)[1 + \Theta]} \right) - 1 \right]^{-1} \approx \hat{f} \left( 1 + \hat{f} e^{q/a T} \frac{q}{a T} \Theta \right) = \hat{f} \left( 1 - \frac{d \ln \hat{f}}{d \ln q} \Theta \right), \quad \varepsilon = - \frac{d \hat{f}}{d \ln q} \Theta,
\]
(5.74)
where we used the relation and the background quantities are
\[
\frac{d \ln \hat{f}}{d \ln q} = - \hat{f} e^{q/a T} \frac{q}{a T}, \quad \bar{\rho} = \frac{1}{3} \bar{\rho} = \frac{4 \pi g}{a^4} \int dq \, q^3 \hat{f}.
\]
(5.75)
Therefore, the perturbation quantities for massless particles are
\[
\delta \rho = \frac{1}{3} \bar{\delta} \rho = \frac{4 \pi g}{a^4} \int_0^\infty dq \ q^3 f_{i0} = 4 \bar{\rho} \ \Theta_{00} \ ,
\]
\[
\pi^{ij} = \frac{g}{a^4} \int dq \ q^3 \int d\Omega \left( n^i n^j - \frac{1}{3} \delta^{ij} \right) \left( - \frac{d\bar{f}}{d\ln q} \Theta \right) = 4 \bar{\rho} \int d\Omega \ 4\pi \left( n^i n^j - \frac{1}{3} \delta^{ij} \right) \left( \frac{1}{3} \delta;_\Theta \right) \Theta = \Pi^{(m)} Q_{\alpha \beta}^{(m)} \ ,
\]
where the useful relations are
\[
\Pi^{(0)} = k^2 \Im \Pi = \frac{4}{3} \bar{\rho} \Theta_{20} \ ,
\]
Finally, noting that
\[
\frac{d}{d\eta} f = f' + n^\alpha \partial_\alpha f = - \frac{d\bar{f}}{d\ln q} \frac{d\Theta}{d\eta} \ ,
\]
we show that the collisionless Boltzmann equation for massless particles becomes
\[
0 = - \frac{d\bar{f}}{d\ln q} \left[ \frac{d\Theta}{d\eta} + \mathcal{A}_\parallel + \left( U - B \right)'_\parallel + \varphi' + n^\alpha n^\beta \left( \mathcal{G}_\alpha' + \mathcal{C}_\alpha' + C_{\alpha \beta}' \right) \right] .
\]
The transformation properties of the temperature anisotropies are
\[
\delta \xi \Theta = - \left( \frac{d\bar{f}}{d\ln q} \right)^{-1} \delta \xi f = \mathcal{H} T , \quad \delta \theta \Theta = - \left( \frac{d\bar{f}}{d\ln q} \right)^{-1} \delta \theta f = - n^i \theta^\eta .
\]
Therefore the fully gauge-invariant temperature anisotropies and the Boltzmann equation are
\[
\Theta_{\eta i} = \Theta + H \chi + V_\parallel , \quad 0 = - \frac{d\bar{f}}{d\ln q} \left[ \frac{d\Theta_{\eta i}}{d\eta} + \alpha_{\chi,\parallel} + \varphi' + n^\alpha n^\beta \left( \mathcal{G}_{\alpha \beta} + C_{\alpha \beta} \right) \right] .
\]
Similarly, the multipole coefficients of the temperature anisotropies transform as
\[
\delta \xi \Theta_{lm} = \frac{\pi g}{a^4 \bar{\rho}} \int dq \ q^3 \delta \xi f_{lm} = \mathcal{H} T \delta_{l0} ,
\]
\[
\delta \theta \Theta_{lm} = \frac{\pi g}{a^4 \bar{\rho}} \int dq \ q^3 \delta \theta f_{lm} = - i \delta_{l1} \theta^\eta \left( \frac{\delta_{m,0} - \delta_{m,1} + \delta_{m,1} - \delta_{m,0}}{\sqrt{2}} \right) .
\]
So, the monopole \( \Theta_{l0} \) changes only under diffeomorphisms and the dipole \( \Theta_{1m} \) changes only under Lorentz transformations, while the higher multipoles are fully gauge-invariant. We construct the fully gauge-invariant variables as
\[
\Theta_{\eta i}^{\text{gi}} = \Theta + H \chi + V_\parallel , \quad \Theta_{00}^{\text{gi}} = \Theta_{00} + H \chi , \quad \Theta_{1m}^{\text{gi}} = \Theta_{1m} + V^{(m)} , \quad \Theta_{l0}^{\text{gi}} = \Theta_{l0} \text{ for } l \geq 2 .
\]
Now, in terms of the multipole coefficients of the temperature anisotropies, the Boltzmann equation becomes
\[
0 = \Theta_{00}^{\text{gi}} + \frac{k}{3} \Theta_{10}^{\text{gi}} + \varphi' + \left( H \chi \right)' + \frac{1}{3} k^2 v_\chi = \Theta_{00}^{\text{gi}} + \frac{k}{3} \Theta_{10}^{\text{gi}} + \varphi' ,
\]
\[
0 = \Theta_{10}^{\text{gi}} + k \left( - \Theta_{00}^{\text{gi}} + \frac{2}{5} \Theta_{20}^{\text{gi}} \right) - k \alpha_\chi - k H \chi + V^{(0)'} = \Theta_{10}^{\text{gi}} - k \Theta_{00}^{\text{gi}} + \frac{2}{5} k \Theta_{20}^{\text{gi}} - k \alpha_\chi ,
\]
\[
0 = \Theta_{20}^{\text{gi}} + k \left( - \frac{2}{3} \Theta_{10} + \frac{3}{7} \Theta_{30} \right) - \frac{2}{3} k^2 v_\chi = \Theta_{20}^{\text{gi}} + k \left( - \frac{2}{3} \Theta_{10} + \frac{3}{7} \Theta_{30} \right) ,
\]
where the derivative term \( d\bar{f}/d\ln q \) is integrated by part to give minus sign. The higher-order multipoles for \( l > 2 \) are again gauge-invariant
\[
0 = \Theta_{l0}^{\text{gi}} + k \left( - \frac{l}{2l - 1} \Theta_{l-1,0} + \frac{l + 1}{2l + 3} \Theta_{l+1,0} \right) = \Theta_{l0}^{\text{gi}} + k \left( - \frac{l}{2l - 1} \Theta_{l-1,0} + \frac{l + 1}{2l + 3} \Theta_{l+1,0} \right) .
\]
5.4 Collisional Term

5.4.1 Thompson Scattering

After the electrons and the positrons annihilate ($T \sim 500$ keV, $t \sim 6$ sec), the Universe cools down, as it expands. The scattering process of our interest in the CMB anisotropy formation is the Thompson scattering of the CMB photons and free electrons:

$$\sigma_T := \frac{8\pi}{3} r_e^2 = 6.651 \times 10^{-25} \text{ cm}^2, \quad r_e := \frac{e^2}{m_e c^2} = 2.818 \times 10^{-13} \text{ cm},$$

(5.90)

where $r_e$ is an effective radius of electrons due to the Coulomb interaction. The Thompson scattering cross-section for protons is smaller by $(m_p/m_e)^2 \approx 4 \times 10^6$, hence negligible.

The Thompson scattering depends on the incoming and outgoing photons, and its expression in the rest-frame of electrons is

$$\frac{d\sigma}{d\Omega} = \frac{3}{16\pi} \sigma_T \left[ 1 + (\hat{n}_{\text{in}} \cdot \hat{n}_{\text{out}})^2 \right].$$

(5.91)

As the Boltzmann equation is written in FRW coordinates, we have to Lorentz boost the electron rest-frame into a FRW frame. Accounting for the change in the frames of the electrons and FRW coordinate and treating the electron velocity at the linear order, we can derive the collisional term (see Dodelson Chapter 4) as

$$C \equiv \frac{d\bar{f}}{d\ln q} \Gamma \left[ \Theta(\hat{n}) - \hat{n} \cdot v_b - \frac{3}{16\pi} \int d^2\hat{n}_{\text{in}} \Theta(\hat{n}_{\text{in}}) \left[ 1 + (\hat{n}_{\text{in}} \cdot \hat{n})^2 \right] \right],$$

(5.92)

where we defined the optical depth to the observer as

$$\tau := \int_{\eta}^{\eta_{\text{out}}} d\eta \, a \bar{n}_e \sigma_T, \quad \Gamma := -\tau' = |\tau'| = a \bar{n}_e \sigma_T.$$

(5.93)

5.4.2 Baryon Evolution

while the Thompson scattering is elastic and conserves energy after collisions in the rest-frame, the energy is transferred in the lab frame. In the absence of collision, the baryons evolve adiabatically, and the thermodynamic equation per unit particle mass is

$$dQ = \frac{3}{2} d \left( \frac{p}{\rho_b} \right) + p_b d \left( \frac{1}{\rho_b} \right) = 0, \quad T_b + 2\dot{H}T_b = dT_e = 0,$$

(5.94)

where we approximated baryons with mono atomic gas $\gamma = 5/3$. With collisions, however, we have to account for the energy injection, and its thermal evolution is modified as

$$Q' = \frac{4\rho_b}{\rho_b} \Gamma k(T_{\gamma} - T_b), \quad dT_e = \frac{2\mu}{m_e R} (T_{\gamma} - T_b),$$

(5.97)

where $\mu$ is the molecular weight, i.e., the number of free particles per atomic mass and we defined the ratio

$$R := \frac{\dot{\rho}_b}{\dot{\rho}_{\gamma}} = \frac{3\dot{\rho}_b}{4\dot{\rho}_{\gamma}} = 0.63 \left( \frac{\omega_b}{0.02} \right) \left( \frac{a}{10^{-3}} \right).$$

(5.98)

The baryon temperature would decrease as $T_b \propto 1/a^2$ in the absence of the energy transfer $dT_e = 0$, but efficient collisions keep the baryon temperature equal to the photon temperature.

---

2The Thompson scattering cross-section for protons is smaller by $(m_p/m_e)^2 \approx 4 \times 10^6$, hence negligible.

3For ideal gas, the internal energy is $U = NkT \times (d_f/2)$, where $d_f$ is the number of degrees of freedom. The adiabatic index $\gamma := c_p/c_v$, the ratio of heat capacity, is related as $\gamma = 1 + 2/d_f$, such that for mono atomic gas $\gamma = 5/3$. So, the internal energy is then $U = NkT/(\gamma - 1) = PV/(\gamma - 1)$. For adiabatic evolution $dQ = 0$, we derive

$$dU = -pdV, \quad p \propto \rho^{-\gamma}, \quad T \propto \rho^{\gamma - 1}.$$
The baryon density evolution is not affected, as the number density is conserved. However, the baryon velocities are also affected by the Thompson scattering of CMB photons as

\[ v_x' + \mathcal{H}v_x = c_s^2 \delta_b + \alpha_x + \frac{\Gamma}{R} \left( \frac{1}{k} \Theta_{10} - v_x \right), \]  

where we used the energy-momentum conservation equation with energy transfer and we defined the sound speed of baryons

\[ c_s^2 = \frac{\dot{\rho}_b}{\rho_b} = \frac{kT_b}{\mu} \left( 1 - \frac{1}{3} \frac{d \ln T_b}{d \ln a} \right), \]  

(5.100)

In general, the baryon pressure is negligible, but its fluctuations need to be considered. Note that the total sound speed of the baryon-photon plasma is

\[ c_b^2 = \frac{\delta p}{\rho} = \frac{1}{3} \frac{1}{1 + R}. \]  

(5.101)

### 5.4.3 Boltzmann Equation for Photons

Using the multipole decomposition, the collisional term can be further manipulated as

\[ C \equiv \frac{df}{d \ln q} \Gamma \left[ \Theta(\hat{n}) - \hat{n} \cdot v_b - \Theta_{00} + \frac{1}{2} P_2(\mu) \left( \frac{1}{5} \Theta_{20} + \frac{1}{5} \Theta_{02}^\parallel + \Theta_{00}^\parallel \right) \right], \]  

(5.102)

and combined with the collisionless Boltzmann equation (5.80), the full Boltzmann equation becomes

\[ \frac{d\Theta}{d\eta} + A_{\|} + (U - B)_{\|} + \varphi' + \alpha' \delta_b = \Gamma \left[ \Theta - \hat{n} \cdot v_b - \Theta_{00} + \frac{1}{2} P_2(\mu) \left( \frac{1}{5} \Theta_{20} + \frac{1}{5} \Theta_{02}^\parallel + \Theta_{00}^\parallel \right) \right], \]  

(5.103)

where we used

\[ \int d\Omega Y_{l0}(\hat{n}) \mu^2 = \sqrt{\frac{4}{15}} \delta_{l2} \delta_{m0} + \sqrt{\frac{1}{3}} \delta_{l0} \delta_{m0}. \]  

(5.104)

When the angular dependence is ignored, we can replace the factor \((3/16\pi)\) with \(1/4\pi\), and we obtain only \(\Theta_{00}\). Applying the multipole decomposition on both sides, we derive

\[ 0 = \Theta_{00}^{\mu} + \frac{k}{3} \Theta_{10}^{\parallel} + \varphi', \quad \Theta_{00} = \frac{1}{4} \delta_{\gamma}, \quad \Theta_{10} = v_{\gamma}' = kv_{\gamma}, \]  

(5.105)

\[ 0 = \Theta_{20}^{\mu} - k \Theta_{00}^{\parallel} + \frac{2}{5} k \Theta_{20}^{\parallel} - k \alpha \chi + \Gamma \left( \Theta_{10} - kv_{\gamma} \right), \]  

(5.106)

\[ 0 = \Theta_{20}^{\parallel} + k \left( \frac{-2}{3} \Theta_{00}^{\parallel} + \frac{3}{7} \Theta_{20}^{\parallel} \right) + \Gamma \left( \frac{-9}{10} \Theta_{20} + \frac{1}{8} \Theta_{00}^{\parallel} + \frac{1}{8} \Theta_{00}^{\parallel} \right). \]  

The higher-order multipoles for \(l > 2\) are again gauge-invariant

\[ 0 = \Theta_{l0} + k \left( -\frac{l}{2l-1} \Theta_{l-1,0} + \frac{l + 1}{2l+3} \Theta_{l+1,0} \right) + \Gamma \Theta_{l0} = \Theta_{l0} + k \left( -\frac{l}{2l-1} \Theta_{l-1,0} + \frac{l + 1}{2l+3} \Theta_{l+1,0} \right) + \Gamma \Theta_{l0}. \]  

In the early Universe, when the collision is efficient \(\Gamma \gg 1\), the higher-multipoles are highly suppressed:

\[ \Theta_{l0}' \sim \frac{\Theta_{l0}}{\eta}, \quad \Gamma \Theta_{l0} \sim \frac{\tau}{\eta} \Theta_{l0}, \quad \therefore \quad \Theta_{l0}' \approx \Gamma \Theta_{l0}, \]  

(5.107)

and from the multipole equations we derive

\[ 0 \sim 0 - k \Theta_{l-10} + 0 + \Gamma \Theta_{l0}, \quad \therefore \quad \Theta_{l0} \sim \frac{k \eta}{\tau} \Theta_{l-10}. \]  

(5.108)
5.5 Initial Conditions for the Evolution

We would like to set up the initial conditions, with which the Boltzmann equations can be evolved in time. At early time in the radiation-dominated era, the set of the Boltzmann-Einstein equations can be greatly simplified on large scales as

\[ 0 = \Theta'_{00} + \varphi'_X, \quad 0 = N'_{00} + \varphi'_X, \quad 0 = \delta'_X + 3\varphi'_X, \quad \therefore \Theta'_{00} = N'_{00} = \frac{1}{3} \delta'_X = -\varphi'_X, \]  

(5.109)

where we used \( k\eta \ll 1 \). Using the adiabatic condition, the initial conditions are set

\[ \Theta_{00} = N_{00} = \frac{1}{3} \delta. \]  

(5.110)

Using the Einstein equation (3.112) in the limit \( k \to 0 \), we derive the constant in the initial conditions

\[ \Theta_{00} = -2\alpha_X, \]  

(5.111)

where we used the adiabatic condition and the rde condition

\[ \bar{\rho}_T \simeq \bar{\rho}_\gamma + \bar{\rho}_\nu, \quad \delta_\gamma = 4\Theta_{00}, \quad \kappa = 3H\alpha_X, \quad \Theta_{10} = kv_\gamma. \]  

(5.112)

Similarly, the Einstein equation (3.110) gives the initial conditions for the velocity scalar

\[ v_\gamma = v_\nu = v_{\text{dm}} = v_b = \frac{\alpha_X}{2H}. \]  

(5.113)

Beyond the monopole and the dipole, neutrinos have small but a non-zero quadrupole moment, arising from its free streaming. Using Eq. (5.78), we derive

\[ 8\pi G\Pi = \frac{8\pi G}{k^2} \Pi^{(0)} = \frac{12}{5} \frac{H^2}{k^2} f_\nu N_{20}, \quad N_{20} = -\frac{5}{12} \frac{k^2}{H^2} \frac{\alpha_X + \varphi_X}{f_\nu}, \]  

(5.114)

where we used

\[ 8\pi G\bar{\rho}_\nu = 3H^2 f_\nu, \quad f_\nu := \frac{\bar{\rho}_\nu}{\bar{\rho}_T}. \]  

(5.115)

Finally, using the evolution equation for \( N_{20} \) and ignoring \( N_{30} \), we have

\[ N'_{20} = \frac{2}{3} kN_{10} = \frac{k^2}{3H} \alpha_X. \]  

(5.116)

Noting that \( H = 1/\eta \) in RDE, we arrive at the relation

\[ N_{20} = \frac{k^2}{6H^2} \alpha_X, \quad 8\pi G\Pi = \frac{2}{5} f_\nu \alpha_X, \quad \varphi_X = -\left(1 + \frac{2}{5} f_\nu \right) \alpha_X. \]  

(5.117)

The comoving-gauge curvature perturbation is generated during the inflationary period, and it is conserved on super horizon scales. After the inflationary period ends, the Universe enters into the standard radiation dominated era (under the assumption that the reheating period is very short). The conformal Newtonian gauge potential is then related to the curvature perturbation on large scales as

\[ \varphi_v = \varphi_X - \frac{1}{2} \left( \alpha_X - \frac{\varphi_X}{H} \right) \simeq \frac{3}{2} \alpha_X - \frac{2}{5} f_\nu \alpha_X, \quad \alpha_X = -\frac{10}{15 + 4f_\nu} \varphi_v, \]  

(5.118)

where we used \( w = 1/3 \) and ignored the time-derivative terms.
6 Weak Gravitational Lensing

6.1 Gravitational Lensing by a Point Mass

In classical mechanics, the gravitational interaction due to a point mass $M$ provides a perturbation along the transverse direction to a test particle moving with the relative speed $v_{rel}$:

$$\Delta v_\perp = \frac{2GM}{b}v_{rel},$$  \hspace{1cm} (6.1)$$

where $G$ is the Newton’s constant and $b$ is the transverse separation (or the impact parameter). The prediction for the light deflection angle $\hat{\alpha}$ in Einstein’s general relativity is well-known to follow the same result in classical mechanics, but with additional factor two:

$$\hat{\alpha} = 4\frac{GM}{bc^2} = 8.155 \times 10^{-3} \text{ arcsec}.$$  \hspace{1cm} (6.2)$$

Given the deflection angle $\hat{\alpha}$, we can readily write down the lens equation in terms of the angular diameter distances

$$D_s \hat{s} = D_s \hat{n} - D_{ls} \hat{\alpha}, \quad \hat{s} = \hat{n} - \theta_E^2/\hat{n},$$  \hspace{1cm} (6.3)$$

where the Einstein radius is

$$\theta_E = \sqrt{\frac{4GM}{c^2}} \frac{D_{ls}}{D_l D_s} = 2.853 \times 10^{-3} \text{ arcsec} \left( \frac{M}{M_\odot} \right) \left( \frac{b}{\text{AU}} \right)^{-1}.$$  \hspace{1cm} (6.4)$$

For a point mass lens and a point source, two lensed image positions are readily obtained as

$$\hat{n}_1 = \frac{1}{2} \left( \hat{s} + \sqrt{\hat{s}^2 + 4\theta_E^2} \right), \quad \hat{n}_2 = \frac{1}{2} \left( \hat{s} - \sqrt{\hat{s}^2 + 4\theta_E^2} \right) < 0, \quad \hat{n}_1 + \hat{n}_2 = \hat{s},$$  \hspace{1cm} (6.5)$$

and when the source and the lens are aligned, the lensed images form a ring with radius $\theta_E$.

- microlensing, probe of MACHOs or exoplanets

6.2 Standard Weak Lensing Formalism

6.2.1 Lens Equation and Distortion Matrix

This light deflection due to a point mass can be generalized to derive the standard weak lensing formalism by considering the gravitational potential fluctuation $\psi = -GM/r$ of the general matter distribution $\rho$ (but still a single lens plane), instead of a point mass ($\psi$ indeed corresponds to the metric fluctuation $\alpha_\chi$). The lensing potential $\Phi$ is the line-of-sight integration of the metric fluctuation,

$$\Phi := \frac{1}{c^2} \frac{D_{ls}}{D_l D_s} \int dz \, 2\psi,$$  \hspace{1cm} (6.6)$$

and using the Poisson equation, we can relate the lensing potential with the surface density $\Sigma$ as

$$\nabla^2 \psi = 4\pi G \bar{\rho} a^2 \delta, \quad \nabla^2 \Phi = 2 \frac{\Sigma}{\Sigma_c}, \quad \Phi(\hat{n}) = \int d^2 \hat{n}' \frac{\Sigma}{\pi \Sigma_c} \ln |\hat{n} - \hat{n}'|,$$  \hspace{1cm} (6.7)$$

where we ignored the boundary term when the Poisson equation is integrated and the critical surface density is defined as

$$\Sigma_c^{-1} := \frac{4\pi G D_{ls} D_l}{c^2}, \quad \Sigma_c = 1.663 \times 10^6 h M_\odot \text{ pc}^{-2} \left( \frac{D_s}{D_{ls}} \right) \left( \frac{D_l}{D_{ls}} \right)^{-1}.$$  \hspace{1cm} (6.8)$$

Though the lensing potential is formally divergent for a point mass, its angular derivative is well defined:

$$\hat{\alpha} = \left( \frac{D_l D_s}{D_{ls}} \right) \nabla_\perp \Phi = \frac{D_s}{D_{ls}} \hat{\nabla} \Phi,$$  \hspace{1cm} (6.9)$$
such that the lens equation becomes
\[ \dot{s} = \dot{n} - \nabla \Phi , \]  
(6.10)
where \( \nabla \) is the angular gradient. When the lensing material is distributed over the redshift, the lensing potential is then obtained by integrating the potential fluctuation over the line-of-sight distance as
\[ \Phi = \int_{0}^{s} d\bar{r} \left( \frac{\bar{r} - \bar{r}}{r\bar{r}} \right) 2\psi \equiv \int_{0}^{s} d\bar{r} \frac{g(\bar{r})}{r^2} 2\psi , \]  
(6.11)
where we switched to a comoving angular diameter distance \( \bar{r} \) and we defined the weight function \( g \) for later convenience
\[ g := \bar{r}^2 \left( \frac{\bar{r} - \bar{r}}{r\bar{r}} \right) . \]  
(6.12)
When the background source galaxies are also spread over some redshift with the distribution \( n_s(r_s) \), the lensing potential can be readily generalized by replacing the weight function with
\[ g := \bar{r}^2 \int_{r}^{\infty} d\bar{r} \left( \frac{\bar{r} - \bar{r}}{r\bar{r}} \right) n_s(\bar{r}_s) , \quad \Phi = \int_{0}^{\infty} d\bar{r} \frac{g(\bar{r})}{r^2} 2\psi , \quad 1 = \int_{0}^{\infty} d\bar{r} \ n_s(\bar{r}_s) , \]  
(6.13)
where the upper limit for the integration is indeed \( \bar{r}(z = \infty) \) and the source distribution is normalized.\(^1\)

Using the lens equation, the distortion matrix \( D \) (or sometimes called the amplification matrix) is defined as
\[ D_{ij} := \frac{\partial s_i}{\partial n_j} = \mathbb{I}_{ij} - \left( \begin{array}{cc} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{array} \right) , \quad \Phi_{ij} := \nabla_j \nabla_i \Phi , \]  
(6.14)
where \( \mathbb{I} \) is the two-dimensional identity matrix and we defined a short hand notation for the angular derivatives of the lensing potential. The distortion matrix is conventionally decomposed into the trace, the traceless symmetric and the anti-symmetric matrices:
\[ D := \mathbb{I} - \left( \begin{array}{cc} \kappa & 0 \\ 0 & \kappa \end{array} \right) - \left( \begin{array}{cc} \gamma_1 & \gamma_2 \\ \gamma_2 & -7_1 \end{array} \right) - \left( \begin{array}{cc} 0 & \omega \\ -\omega & 0 \end{array} \right) , \quad \det D = (1 - \kappa)^2 - \gamma^2 + \omega^2 , \]  
(6.15)
where the trace is the gravitational lensing convergence \( \kappa \) and the symmetric traceless part is the lensing shear \( \gamma = \sqrt{\gamma_1^2 + \gamma_2^2} : \)
\[ \kappa \equiv 1 - \frac{1}{2} \text{Tr} D = \frac{1}{2} (\Phi_{11} + \Phi_{22}) , \quad \omega \equiv \frac{D_{21} - D_{12}}{2} = 0 , \]  
(6.16)
\[ \gamma_1 \equiv \frac{D_{22} - D_{11}}{2} = \frac{1}{2} (\Phi_{11} - \Phi_{22}) , \quad \gamma_2 \equiv -\frac{D_{12} + D_{21}}{2} = \Phi_{12} = \Phi_{21} . \]  
(6.17)
Since the distortion matrix in Eq. (6.14) is symmetric, the rotation \( \omega \) vanishes in the standard formalism at all orders.

The standard lensing formalism is based on the lens equation and the lensing potential in Eq. (6.10). However, the source angular position \( \dot{s} := (\theta + \delta \theta, \phi + \delta \phi) \) is gauge-dependent, and the lensing potential that is responsible for the angular distortion \( (\delta \theta, \delta \phi) \) is also gauge-dependent. Indeed, we already know that \( 2\psi \) in Eq. (6.10) should be \( (\alpha \chi - \varphi \chi) \) to match the leading terms for \( \delta \theta \) in Eq. (6.104) and the Poisson equation in Eq. (6.21) is indeed an Einstein equation with \( \psi \) there replaced by \( -\varphi \chi \).\(^2\) Furthermore, there exist no contributions from the vector and the tensor perturbations in the standard lensing formalism. Finally, while the derivations in this subsection assume no linearity, all formulas of the standard lensing formalism turn out to be valid only at the linear order in perturbations.

6.2.2 Convergence and Shear

While the distortion matrix is defined in terms of angles, it is often assumed in literature that the line-of-sight direction is along \( z \)-axis \((\hat{n} \parallel \hat{z}, \text{i.e., } \theta = 0)\), and two angles are aligned with \( x-y \) plane. In such a setting, consider two small angular vectors at the source position subtended respectively by \( d\theta \) and \( d\phi \) at the observer position
\[ \Delta s^{d\theta}_i = D_{i1} d\theta , \quad \Delta s^{d\phi}_i = D_{i2} d\phi . \]  
(6.18)
\(^1\)Sometimes it is normalized when integrated over redshift.
\(^2\)Additional condition of a vanishing anisotropic pressure is needed to guarantee \( \alpha \chi = -\varphi \chi \) and hence the consistency in the lensing equation.
The solid angle at the source subtended by these two angular vectors is then related to the solid angle at the observer as

\[ d\Omega_s = \left| \Delta s^{d\theta} \times \Delta s^{d\phi} \right| = \det \Phi \ d\theta d\phi = \det \Phi \ d\Omega_o , \]  

(6.19)

and hence the gravitational lensing magnification \( \mu \) is then

\[ \mu^{-1} = \frac{d\Omega_s}{d\Omega_o} = \det \Phi . \]  

(6.20)

For this reason, the distortion matrix is often called the inverse magnification matrix. Using the Poisson equation in cosmology,

\[ \nabla^2 \psi = 4\pi G \bar{\rho} a^2 \delta_m = \frac{3H^2}{2} \Omega_m \frac{\delta_m}{a} , \]  

(6.21)

the gravitational lensing convergence can be computed in terms of the matter density fluctuation \( \delta_m \) in the comoving gauge as

\[ \kappa = \int_0^{r_s} d\bar{r} \ g(\bar{r}, \bar{r}_s) \nabla^2 \psi = \frac{3H^2}{2} \Omega_m \int_0^{r_s} d\bar{r} \ \frac{g(\bar{r}, \bar{r}_s)}{a} \delta_m \]  

(6.22)

where we used

\[ \nabla^2 = \nabla^2_\perp + \frac{\partial^2}{\partial \bar{r}^2} + \frac{2}{\bar{r}} \frac{\partial}{\partial \bar{r}} , \quad \nabla^2_\perp = \frac{1}{\bar{r}^2} \nabla^2 , \]  

(6.23)

and ignored the boundary terms.

In general, we can compute the individual components of the distortion matrix

\[ \gamma_1 = \int_0^{r_s} d\bar{r} \ g(\bar{r}, \bar{r}_s) \left( \nabla^2_1 - \nabla^2_2 \right) \psi , \quad \gamma_2 = 2 \int_0^{r_s} d\bar{r} \ g(\bar{r}, \bar{r}_s) \nabla_1 \nabla_2 \psi , \]  

(6.24)

by using

\[ \frac{1}{2} \Phi_{ij} = \frac{1}{2} \nabla_i \nabla_j \Phi = \int_0^{r_s} d\bar{r} \ g(\bar{r}, \bar{r}_s) \nabla_i \nabla_j \psi , \]  

(6.25)

where the indices \( i, j \) represent the perpendicular components.

### 6.2.3 Angular Power Spectrum and Angular Correlation

Assuming that the survey area is small, we will utilize the angular Fourier transformation in Eq. (2.17) by again computing

\[ \Phi(l) = \int d^2 \theta \ e^{-il \cdot \theta} \int_0^{r_s} d\bar{r} \ \frac{g(\bar{r}, \bar{r}_s)}{\bar{r}^2} \ 2\psi = \int_0^{r_s} d\bar{r} \ \frac{g(\bar{r}, \bar{r}_s)}{\bar{r}^4} \int \frac{dk}{2\pi} \ 2\psi \left( k_\perp = \frac{l}{\bar{r}} \right) e^{ik_\parallel \bar{r}} , \]  

(6.26)

such that the lensing observables are

\[ \kappa(l) = -\frac{l^2}{2} \Phi(l) , \quad \gamma_1(l) = -\frac{l^2 - l_2^2}{2} \Phi(l) = \cos 2\phi_l \kappa(l) , \quad \gamma_2(l) = -l_1 l_2 \Phi(l) = \sin 2\phi_l \kappa(l) , \]  

(6.27)

where we used \( l = (l_1, l_2) = l(\cos \phi_l, \sin \phi_l) \). Therefore, the angular power spectra can be readily derived as

\[ P_\kappa(l) = \frac{l^4}{4} P_\Phi(l) , \quad P_{\gamma_1} = \cos^2 2\phi_l \ P_\kappa(l) , \quad P_{\gamma_2} = \sin^2 2\phi_l \ P_\kappa(l) , \]  

(6.28)

where the angular power spectrum of the lensing potential is obtained by using the Limber approximation as

\[ P_\Phi(l) = \int_0^{r_s} d\bar{r} \ \frac{g^2(\bar{r}, \bar{r}_s)}{\bar{r}^6} 4P\psi \left( k_\perp = \frac{l}{\bar{r}} \right) . \]  

(6.29)

From the relation of the lensing observables, we find it useful to construct E and B-modes as

\[ E(l) := \cos 2\phi_l \ \gamma_1(l) + \sin 2\phi_l \ \gamma_2(l) , \quad B(l) := -\sin 2\phi_l \ \gamma_1(l) + \cos 2\phi_l \ \gamma_2(l) . \]  

(6.30)
We can readily derive
\[ E(l) = \kappa(l) , \quad B(l) = 0 , \quad P_E(l) = P_\kappa(l) , \quad P_B(l) = P_{EB}(l) = 0 , \]  
(6.31)
in the absence of any systematics and/or physics other than the gravitational lensing, such that it provides a consistency check of the measurements, where the convergence power spectrum is again related to the matter power spectrum as
\[ P_\kappa(l) = \left( \frac{3H_0^2}{2\Omega_m} \right)^2 \int_0^{r_s} \frac{dr}{r^2 a^2} P_m \left( k_{\perp} \frac{1}{r} \right) . \]
(6.32)

Now we compute the angular correlation function by Fourier transforming the angular power spectrum. Out of two shear components, we construct three angular correlation functions as
\[ w_{ij}(\theta) := \langle \gamma_i(0) \gamma_j(\theta) \rangle = \int \frac{d^2 l}{(2\pi)^2} e^{i 2 \theta} \left( \begin{array}{c} \cos^2 2\phi_l \\ \cos 2\phi_l \sin 2\phi_l \\ \sin^2 2\phi_l \end{array} \right) P_\kappa(l) \]
(6.33)
\[ = \frac{1}{2} \int_0^{\infty} dl \frac{l}{2\pi} P_\kappa(l) \left( \begin{array}{cc} J_0(l\theta) + J_4(l\theta) & 0 \\ 0 & J_0(l\theta) - J_4(l\theta) \end{array} \right) , \]
(6.34)
where \( J_n \) is the Bessel function and used its integral representation
\[ J_0(x) = \int \frac{d\phi}{2\pi} e^{ix \cos \phi} , \quad J_4(x) = \int \frac{d\phi}{2\pi} e^{ix \cos \phi} \cos 4\phi . \]
(6.35)

### 6.2.4 Worked Examples

For the simplest case, where the lens and the source are at two definite redshift slices, the lensing observables can be written in a polar coordinate as
\[ 2\kappa = \Phi_{rr} + \frac{\Phi_r}{r} + \frac{\Phi_{\theta\theta}}{r^2} = 2 \frac{\Sigma}{\Sigma_c} , \]
(6.36)
\[ 2\gamma_1 = \cos 2\theta \Phi_{rr} - 2 \frac{\sin 2\theta}{r} \Phi_r - \cos 2\theta \Phi_\theta + \frac{\cos \theta}{r^2} \Phi_{r\theta} + 2 \frac{\sin \theta}{r^2} \Phi_\theta , \]
(6.37)
\[ 2\gamma_2 = \sin 2\theta \Phi_{rr} + 2 \frac{\cos 2\theta}{r} \Phi_r - \sin 2\theta \Phi_\theta - \frac{\sin \theta}{r^2} \Phi_{r\theta} - 2 \frac{\cos \theta}{r^2} \Phi_\theta , \]
(6.38)
\[ \gamma^2 := \gamma_1^2 + \gamma_2^2 = \frac{1}{4} \left( \Phi_{rr} - \Phi_r \frac{\Phi_{\theta\theta}}{r^2} \right)^2 + \left( \frac{\Phi_r}{r} - \frac{\Phi_{\theta\theta}}{r^2} \right)^2 . \]
(6.39)

For an axisymmetric lens, the lensing observables are further simplified, and the convergence and shear are
\[ 2\kappa = \nabla^2 \Phi = \Phi_{rr} + \frac{\Phi_r}{r} = 2 \frac{\Sigma}{\Sigma_c} , \quad \gamma = \frac{1}{2} \left( \Phi_{rr} - \Phi_r \right) = \frac{\Phi_r}{r} - \frac{\Sigma}{\Sigma_c} = \frac{\Sigma(\langle r \rangle) - \Sigma}{\Sigma_c} , \]
(6.40)
where \( \Sigma(\langle r \rangle) \) is the average surface density enclosed in radius \( r \) and \( \Sigma(\langle r \rangle) = \Phi_r/r \) from the first relation. The magnification is determined by the surface density of the lensing material, and the gravitational shear is set by the excess surface density of the enclosed mass \( \Delta \Sigma := \Sigma(\langle r \rangle) - \Sigma(r) \).

For a point mass, the convergence and the shear are
\[ \psi = -\frac{GM}{r} , \quad \nabla^2 \psi = 4\pi GM \delta^D(x) , \quad \kappa = \frac{\Sigma}{\Sigma_c} = \frac{M \delta^D(R)}{\Sigma_c} , \]
(6.41)
\[ \gamma = \frac{\Sigma(\langle R \rangle) - \Sigma}{\Sigma_c} = \frac{\Sigma}{\Sigma_c} = \frac{\theta_E^2}{2} , \quad \Sigma := \frac{M}{\pi R^2} . \]
(6.42)

The lensing magnification is then
\[ \mu^{-1} = \det \mathbb{D} = 1 - \Phi_{rr} - \frac{\Phi_r}{r} + \Phi_r \frac{\Phi_r}{r} = \frac{s \partial_s}{r \partial r} , \]
(6.43)
where we used the lens equation
\[ s = r - \Phi_r , \quad \partial_r s = 1 - \Phi_{rr} . \]
(6.44)
For a point mass, there exist two lensed images. When two images are not spatially resolved, the magnification of the lensed images is the sum of two, and we derive the master equation for microlensing
\[ \mu = \left( \frac{s \partial_s}{r \partial r} \right)^{-1} + \left( \frac{s \partial_s}{r \partial r} \right)^{-1} = \frac{u^2 + 2}{u \sqrt{u^2 + 4}} , \quad u := \frac{s}{\theta_E} . \]
(6.45)
6.2.5  Galaxy-Galaxy Lensing

Galaxy-galaxy lensing is used to refer to the two-point correlation of the galaxies at one point and the lensing signal measured by background galaxies at the other point. In short, it measures the galaxy-matter cross-correlation. Compared to the cosmic shear measurements, the advantage here is that we have well-defined lenses (lens galaxies) in the foreground, such that the shear measurements in galaxy-galaxy lensing are less susceptible to other systematics.

Assuming spherical symmetry, we can readily derive the lensing convergence and the shear as

\[ \kappa(\theta) = \frac{\Sigma(\theta)}{\Sigma_c}, \quad \gamma(\theta) = \bar{\kappa}(\theta) - \kappa(\theta), \]  

(6.46)

where the “comoving” critical surface density is\(^3\)

\[ \Sigma_c(z_1, z_2) = \frac{c^2}{4\pi G} \frac{r_s}{r_l r_{ls}} \frac{1}{1 + z_l} = 1.663 \times 10^{18} \ h Mpc^{-2} \ r_s \ \left( \frac{r_{ls}}{h^{-1} \ Mpc} \right)^{-1} \ \frac{1}{1 + z_l}. \]

(6.47)

Since we are measuring the excess matter around galaxies, the lensing observables are related to the projected galaxy-matter correlation function:

\[ w(R) = \int_{-\infty}^{\infty} dz \ \xi_{gm} \left( r = \sqrt{R^2 + z^2} \right) = \int_0^{\infty} \frac{dk_{\perp}}{2\pi} \ P_{gm}(k_{\perp}) \ J_0(k_{\perp}R), \]

(6.48)

where the integration along the line-of-sight is performed. However, note that this is valid only on small angle, as the observed angular separation \( \theta \), not the physical separation \( R \) is kept fixed. Under the small-angle approximation, the convergence at a given separation can be derived as

\[ \kappa(\theta) = \frac{\Sigma(\theta)}{\Sigma_c} = \int dz \ \frac{\bar{\rho}_m}{\Sigma_c} \ (1 + \xi_{gm}) = \frac{\bar{\rho}_m}{\Sigma_c} \ w(R), \]

(6.49)

and the remaining lensing observables are

\[ \bar{\kappa}(\theta) = \frac{2\pi}{\theta^2} \int_0^{\theta} d\theta \ \kappa(\theta) = \frac{3H_0^2 \Omega_m}{2} \frac{(\bar{r} - \bar{r}_l)}{a_l r_s} \int \frac{dk_{\perp}}{2\pi} \ P_{gm}(k_{\perp}; \bar{r}_l) \ J_1(k_{\perp}\bar{r}) \]

\[ - \int \frac{dk_{\parallel}}{2\pi} \ P_{gm}(k_{\parallel}; \bar{r}_l) \ J_2(k_{\parallel}\bar{r}), \]

\[ \Delta \Sigma(R) = \Sigma_c \gamma_T = \bar{\rho}_m \int_{-\infty}^{\infty} dz \left[ \frac{2}{R^2} \int_0^R dR' \ R' \ \xi_{gm}(R', z) - \xi_{gm}(R, z) \right], \]

(6.50)

\[ \gamma_T(\theta) = \bar{\kappa}(\theta) - \kappa(\theta) = \frac{3H_0^2 \Omega_m}{2} \frac{(\bar{r} - \bar{r}_l)}{a_l r_s} \int \frac{dk_{\perp}}{2\pi} \ P_{gm}(k_{\perp}; \bar{r}_l) \ J_2(k_{\perp}\bar{r}) \]

\[ - \int \frac{dk_{\parallel}}{2\pi} \ P_{gm}(k_{\parallel}; \bar{r}_l) \ J_2(k_{\parallel}\bar{r}), \]

\[ \Delta \Sigma(R) = \Sigma_c \gamma_T = \bar{\rho}_m \int_{-\infty}^{\infty} dz \left[ \frac{2}{R^2} \int_0^R dR' \ R' \ \xi_{gm}(R', z) - \xi_{gm}(R, z) \right]. \]

(6.51)

6.3  Weak Lensing Observables

6.3.1  Ellipticity of Galaxies

The ellipticity \( \epsilon \) of galaxies is measured in terms of its semi-major axis \( a \) and the semi-minor axis \( b \) or in terms of the axis ratio \( q \) as

\[ \epsilon := \frac{a^2 - b^2}{a^2 + b^2} = \frac{1 - q^2}{1 + q^2} \approx \frac{\delta - \frac{1}{2} \delta^2}{1 - \delta + \frac{1}{2} \delta^2} \approx \delta; \quad q := \frac{b}{a} := 1 - \delta. \]

(6.53)

In an idealized case of round galaxies, the ellipticity \( \epsilon \), the axis ratio \( q \), and the distortion \( \delta \) are a measure of gravitational lensing effects of intervening matter, and they are equivalent in the weak lensing regime. In observations, the center of the galaxy and its ellipticity moment are measured by using some weight function \( W[L_r(\hat{n})] \) of the observed intensity as

\[ \hat{n}_o := \int \frac{d^2 \hat{n}}{d^2 \hat{n}} \ W[\hat{n}], \quad \mathcal{M}_{ij} := \int \frac{d^2 \hat{n}}{d^2 \hat{n}} \ W[\hat{n}] \]

(6.54)

\(^3\)Multiply by \((1 + z_l)\)^2 for physical critical surface density. Note that sometimes people use the physical angular diameter distances, while using comoving coordinates for other quantities, in which \((1 + z_l)\)^2 appears in the equation, instead of \((1 + z_l)\).
where the simplest weight function is just the observed intensity $W = I[\hat{n}]$. Given the ellipticity moment, we can define the ellipticity vector and the position angle as

$$\epsilon := \left( \frac{M_{xx} - M_{yy}}{M_{xx} + M_{yy}}, \frac{2M_{xy}}{M_{xx} + M_{yy}} \right) := (\epsilon_+, \epsilon_x) = (\cos 2\Theta, \sin 2\Theta), \quad \tan 2\Theta \equiv \frac{2M_{xy}}{M_{xx} - M_{yy}}. \quad (6.55)$$

Note that the ellipticity vector is headless, such that it is identical under 180 degree rotation, or spin 2. Since only the ellipticity vector matters, the ellipticity moments $M$ are often defined without the denominator.

### 6.3.2 Lensing Polarization

The ellipticity moments of the source galaxies would be what we measure in the absence of gravitational lensing. However, the gravitational lensing changes the observed ellipticity moments. Now, for simplicity, we will ignore rotation ($\omega = 0$) and express the distortion matrix in our coordinate:

$$D = I - \begin{pmatrix} \kappa & 0 \\ 0 & \kappa \end{pmatrix} - \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_2 & -\gamma_1 \end{pmatrix}, \quad (6.56)$$

and the magnification matrix is then the inverse of the distortion matrix:

$$M_{ij} := D_{ij}^{-1} = \frac{1}{|D|} \begin{pmatrix} 1 - \kappa + \gamma_1 & \gamma_2 \\ \gamma_2 & 1 - \kappa - \gamma_1 \end{pmatrix}, \quad \mu := |M| = |D|^{-1} = \frac{1}{(1-\kappa)^2 - \gamma_1^2}. \quad (6.57)$$

Further, assuming the surface brightness conservation due to gravitational lensing (i.e., no frequency change), the observed ellipticity moments are related to those in the source rest-frame as

$$M_{ij}^s := \int d^2\hat{n} \hat{n}_i \hat{n}_j W[\hat{n}] = \int |M| d^2\hat{s} \ M_{ik}\hat{s}_k \ M_{jl}\hat{s}_l \ W[\hat{s}] \simeq \mu \ M_{ik}\ M_{jl}\ M_{kl}^s, \quad (6.58)$$

where we assumed $\hat{n}_s = 0$ and the source size is small that the magnification matrix is constant over the area. Using the definition of the source ellipticity moments

$$M_{11}^s = \frac{1 + \epsilon_+^s}{2} M, \quad M_{22}^s = \frac{1 - \epsilon_+^s}{2} M, \quad M_{12}^s = \epsilon_x^s M, \quad M := M_{11}^s + M_{22}^s, \quad (6.59)$$

the observed ellipticity can be derived in terms of the magnification matrix as

$$\epsilon_+^l = \frac{(1 + \epsilon_+^s)M_{11}^s + 2\epsilon_x^s M_{12}^s (M_{11}^s - M_{22}^s) - 2\epsilon_x^s M_{22}^s (1 - \epsilon_+^s)M_{22}^s}{(1 + \epsilon_+^s)M_{11}^s + 2\epsilon_x^s M_{12}^s (M_{11}^s + M_{22}^s) + 2\epsilon_x^s M_{22}^s (1 - \epsilon_+^s)M_{12}^s + (1 - \epsilon_+^s)M_{22}^s}, \quad (6.60)$$

$$\epsilon_x^l = \frac{2M_{12}^s [\epsilon_x^s M_{12}^s + (1 - \epsilon_+^s)M_{22}^s] + 2M_{11}^s [1 + \epsilon_+^s)M_{12}^s + \epsilon_x^s M_{22}^s]}{(1 + \epsilon_+^s)M_{11}^s + 2\epsilon_x^s M_{12}^s (M_{11}^s + M_{22}^s) + 2M_{12}^s + (1 - \epsilon_+^s)M_{22}^s}, \quad (6.61)$$

where the relation is exact. In terms of the lensing convergence and shear, we derive

$$\epsilon_+^l = \frac{\epsilon_+^s [(1 - \kappa)^2 + \gamma_1^2 - \gamma_2^2] + 2\epsilon_x^s \gamma_1 \gamma_2 + 2\gamma_1 (1 - \kappa)}{2\epsilon_x^s \gamma_1 (1 - \kappa) + 2\epsilon_x^s \gamma_2 (1 - \kappa) + (1 - \kappa)^2 + \gamma_1^2 + \gamma_2^2}, \quad (6.62)$$

$$\epsilon_x^l = \frac{\epsilon_x^s [(1 - \kappa)^2 + \gamma_1^2 - \gamma_2^2] + 2\epsilon_x^s \gamma_1 \gamma_2 + 2\gamma_2 (1 - \kappa)}{2\epsilon_x^s \gamma_1 (1 - \kappa) + 2\epsilon_x^s \gamma_2 (1 - \kappa) + (1 - \kappa)^2 + \gamma_1^2 + \gamma_2^2}. \quad (6.63)$$

For circular sources, where $M_{11}^s = M_{22}^s \neq 0$, and $M_{12} = 0$ (or $\epsilon_+^s = \epsilon_x^s = 0$), the observed ellipticity becomes

$$\epsilon_+^l = \frac{2\gamma_1 (1 - \kappa)}{(1 - \kappa)^2 + \gamma_1^2 + \gamma_2^2} \simeq 2\gamma_1, \quad \epsilon_x^l = \frac{2\gamma_2 (1 - \kappa)}{(1 - \kappa)^2 + \gamma_1^2 + \gamma_2^2} \simeq 2\gamma_2, \quad (6.64)$$

where we expanded to the linear order in the last step. Assuming there is no ellipticity correlation of the source galaxies

$$\langle \epsilon_+ \rangle = \langle \epsilon_x \rangle = \langle \epsilon_+ \epsilon_x \rangle = \frac{1}{2} \langle \epsilon^2 \rangle \langle \sin 4\phi \rangle = 0, \quad (6.65)$$

$$\langle \epsilon_+^2 \rangle = \langle \epsilon_x^2 \rangle = \langle \epsilon^2 \rangle \langle \cos^2 2\phi \rangle = \frac{1}{2} \langle \epsilon^2 \rangle, \quad (6.66)$$

\(^4\)With rotation, the magnification matrix is not symmetric, $M_{12} \neq M_{21}$.  

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Therefore, we fix the normalization factor $C$; it appears natural to choose the normalization to fix the combination $C$ transformed photon wavevector can be parametrized as

$$C_{\mu} \equiv \frac{1}{a^2} g_{\mu\nu} = - (1 + 2A) d\eta^2 - 2A_d x^\nu d\eta + \left[ (1 + 2\varphi)\delta_{\alpha\beta} + 2\gamma_{\alpha\beta} + 2C_{(\alpha,\beta)} + 2C_{\alpha\beta} \right] dx^\alpha dx^\beta.$$ (6.69)

Since the null geodesic path ($ds^2 = 0$) remains unaffected by the conformal transformation, we can utilize the geodesic equation in the conformally transformed metric to derive the null path $x^\mu (\Lambda)$. With the conformal transformation, the geometry of the spacetime manifold changes, and the covariant derivatives in two different manifolds are not identical in order to satisfy their own metric compatibility:

$$0 = \nabla_\rho \bar{g}_{\mu\nu}, \quad 0 = \hat{\nabla}_\rho \bar{g}_{\mu\nu},$$ (6.70)

where quantities in the conformally transformed metric are represented with hat. The metric compatibility condition in two manifolds implies Wald (1984) that the covariant derivatives are related to each other with the connecting tensor as

$$\hat{\nabla}_\nu k^\mu = \nabla_\nu k^\mu + C^\mu_{\nu\rho} k^\rho, \quad C^\mu_{\nu\rho} \equiv \mathcal{H} \left( g_{\nu\rho} \delta^\mu \eta - \delta^\mu \delta^\eta \eta - \delta^\mu \delta^\eta \eta \right),$$ (6.71)

Hence, the geodesic equation is not satisfied for the photon wavevector $k^\mu$ with $\hat{\nabla}_\mu$ in the conformally transformed metric. However, by re-parameterizing the photon path $x^\mu (\Lambda)$ with different affine parameter $\lambda$ (instead of $\Lambda$), we can derive the conformally transformed wavevector $\hat{k}^\mu$ for the same null path that satisfies the geodesic equation $0 = \hat{k}^\mu \hat{\nabla}_\mu \hat{k}^\mu$ in the conformally transformed metric:

$$\hat{k}^\mu = \frac{dx^\mu}{d\lambda} = \mathcal{C} a^2 k^\mu, \quad \frac{d\Lambda}{d\lambda} = \mathcal{C} a^2,$$ (6.72)

where the proportionality constant $\mathcal{C}$ is left unconstrained in the conformal transformation, because the metric compatibility constrains only the derivative of $d\Lambda/d\lambda$ Wald (1984).

Given the conformal transformation, the choice of the normalization $\mathcal{C}$ is completely free. With Eqs. (6.72) and (6.78), it appears natural to choose the normalization to fix the combination $\mathcal{C} a \omega$ that is constant everywhere in the background. Therefore, we fix the normalization factor $\mathcal{C}$ by setting the product at the observer position

$$1 \equiv \mathcal{C} a \omega \text{ at } x^\mu (\lambda_o) = x^\mu_o,$$ (6.73)

where the subscript $o$ represents the observer position. The presence of perturbations makes the combination $\mathcal{C} a \omega$ vary as a function of position, while the normalization constant $\mathcal{C}$ is still a constant. With such condition, the conformally transformed photon wavevector can be parametrized as

$$\hat{k}^\mu := (1 + \delta \nu, -n^i \delta^\alpha_i - \delta n^\alpha),$$ (6.74)
and the four velocity in the conformally transformed metric is
\[ \dot{u} = au^\mu, \quad \dot{u}_\mu = \dot{g}_{\mu\nu}u^\nu = \frac{u_\mu}{a}. \]  
(6.75)

We define the perturbation \( \tilde{\Delta} \nu \) in the observed frequency in the conformally transformed metric, in terms of the product
\[ C a \omega = -C a (u_\mu k^\mu) = -\dot{u}_\mu \dot{k}^\mu = 1 + \delta \nu + A + (U_\alpha - B_\alpha) n^\alpha := 1 + \tilde{\Delta} \nu. \]  
(6.76)

### 6.4.2 Boundary Condition for Photon Wavevector

The photon wavevector is measured in the observer rest-frame as
\[ k^\mu = e_\mu^a k^a = (\omega, \mathbf{k}) = \omega(1, -\mathbf{n}), \quad \omega = |\mathbf{k}|, \quad |\mathbf{n}| = 1, \]  
(6.77)
where we expressed the components of the photon wavevector in the observer rest-frame, in terms of the observable quantities: the angular frequency \( \omega = 2\pi \nu \) of the photon and the angular position \( \mathbf{n} \) of the source. In the observer rest-frame, a set of angles \((\theta, \phi)\) is assigned to the unit directional vector \( n^i = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta) \). Using the tetrad expression in Eq. (5.4), we can derive the photon wavevector in a FRW coordinate
\[ k^\mu = e_\mu^a k^a = \frac{\omega}{a} \left[ 1 - \mathcal{A} - n^i \delta^\beta_i (U_\beta - B_\beta)_o - n^i \delta^\alpha_i + U_o + n^i \delta^\beta_i \left( \varphi \delta^\alpha_\beta + \Omega_\alpha,\beta + C^\alpha_\beta \right) + \epsilon^{\alpha ij} n^i \Omega_o \right]. \]  
(6.78)

While the null path \( x^\mu(\Lambda) \) can be parametrized by any affine parameter \( \Lambda \), we can physically fix the affine parameter \( \Lambda \) by demanding that the tangent vector along the path is the photon wavevector,
\[ k^\mu(\Lambda) = \frac{dx^\mu}{d\Lambda}, \]  
(6.79)
and Eq. (6.78) is satisfied at the observer position as the boundary condition for the photon wavevector. Therefore, the perturbations to the photon wavevector at the observer position are then
\[ \delta \nu_o = \tilde{\Delta} \nu_o - \mathcal{A}_o - n^i \delta^\beta_i (U_\beta - B_\beta)_o = \tilde{\Delta} \nu_o - \left[ \alpha_\chi + V_\parallel + \frac{d}{d\Lambda} \left( \frac{\chi}{a} \right) \right] + \mathcal{H} \chi \]  
(6.80)
\[ \delta n^\alpha_o = n^i \delta^\alpha_i \tilde{\Delta} \nu_o - U_o - n^i \delta^\alpha_i \left( \varphi \delta^\alpha_\beta + \Omega_\alpha,\beta + C^\alpha_\beta \right) - \epsilon^{\alpha ij} n^i \Omega_o \]  
(6.81)
\[ = n^i \delta^\alpha_i \left( \tilde{\Delta} \nu - \varphi \chi - \mathcal{H} \chi \right) + - V_o - \Psi_o - C^\alpha_\beta \delta^\alpha_i n^i - \epsilon^{\alpha ij} n^i \Omega_o + \frac{d}{d\Lambda} \Omega_{\alpha} \]  
(6.82)

### 6.4.3 Geodesic Equation and Observed Redshift

The photon wavevector in the conformally transformed metric trivially satisfies the geodesic equation in a homogeneous universe. In the presence of perturbations, the perturbations \((\delta \nu, \delta n^\alpha)\) to the photon wavevector \( \dot{k}_\mu \) are constrained by the temporal and the spatial geodesic equations
\[ 0 = \dot{k}^\mu \nabla_\mu \dot{k}^\alpha := \frac{d}{d\Lambda} \delta \nu + \delta \tilde{\Gamma}^\alpha, \quad 0 = \dot{k}^\mu \nabla_\mu \dot{k}^\alpha := -\frac{d}{d\Lambda} \delta n^\alpha + \delta \tilde{\Gamma}^\alpha, \]  
(6.82)
where we defined the derivative along the photon path with respect to the affine parameter
\[ \frac{d}{d\Lambda} \equiv \dot{k}^\mu \frac{\partial}{\partial x^\alpha} = \left( \frac{\partial}{\partial \eta} - n^\alpha \frac{\partial}{\partial x^\alpha} \right) + \left( \frac{\partial \nu}{\partial \eta} - n^\alpha \frac{\partial}{\partial x^\alpha} \right), \]  
(6.83)
and the perturbations in the geodesic equations
\[ \delta \tilde{\Gamma}^\eta \equiv \dot{\tilde{\Gamma}}^\mu_{\nu \beta} \dot{k}^\nu \dot{k}^\beta = A' - 2A_\alpha n^\alpha + (B_{\alpha \beta} + C_{\alpha \beta}') n^\alpha n^\beta \]  
(6.84)
\[ = \frac{d}{d\Lambda} \left[ 2\alpha_\chi + 2H \chi + \frac{d}{d\Lambda} \left( \frac{\chi}{a} \right) \right] - (\alpha_\chi - \varphi \chi)' + \left( \Psi_{\alpha,\beta} + C_{\alpha \beta}' \right) n^\alpha n^\beta, \]  
\[ \delta \tilde{\Gamma}^\alpha \equiv \dot{\tilde{\Gamma}}^\mu_{\nu \beta} \dot{k}^\nu \dot{k}^\beta = A^\alpha - B^\alpha \chi - (B_{\beta \gamma} - B_{\gamma \beta} + 2C_{\beta \gamma}') n^\beta n^\gamma + \left( 2C_{\beta \gamma}' - C_{\beta \gamma} \right) n^\beta n^\gamma \]  
(6.85)
\[ = (\alpha_\chi - \varphi \chi) n^\alpha - \Psi_{\beta,\gamma} n^\beta - C_{\beta \gamma} n^\beta n^\gamma - \frac{d}{d\Lambda} \left( 2\varphi \chi n^\alpha + \Psi_{\beta,\gamma} + 2C_{\beta \gamma}' + 2H \chi n^\alpha \right) + \frac{d^2}{d\Lambda^2} \Omega, \]  
(6.86)

\(^3\)Note that not all tangent vectors of a given path correspond to the photon wavevector. So, the condition that the tangent vector in Eq. (6.79) is the photon wavevector completely fixes the parametrization of the path.
Note that we already chose a rectangular coordinate for our flat FRW coordinate, in which the Christoffel symbols vanish in the background. In addition to the geodesic equation, the perturbations to the photon wavevector are subject to the null condition $0 = k^a \dot{k}_a$:

$$
n^a \delta n_a = \delta \nu + A - B \parallel - C \parallel , \quad C \parallel := C_{\alpha\beta} n^\alpha n^\beta . \tag{6.86}
$$

With the explicit expressions of the geodesic equations, we integrate them over the affine parameter to obtain the perturbations $(\delta \nu, \delta n^\gamma)_{\lambda}$ along the photon path $x^\lambda$:

$$
\delta \nu_{\lambda} - \delta \nu_o = - \int_0^\lambda d\lambda' \delta \hat{\Gamma}^n = - \left[ 2\alpha + \frac{d}{d\lambda} \left( \frac{\lambda'}{a} \right) \right]_\lambda + \int_0^\lambda d\lambda' \left[ (\alpha - \varphi') - (\Psi_{\alpha,\beta} + C_{\alpha\beta}') n^\alpha n^\beta \right] 
= - \left[ 2\alpha - \Psi + 2H\chi + \frac{d}{d\lambda} \left( \frac{\lambda'}{a} \right) \right]_\lambda - \int_0^{\hat{\nu}_s} d\hat{r} (\alpha - \varphi - \Psi - C_{\parallel})',
$$

and

$$
\delta n^\alpha_{\lambda} - \delta n^\alpha_o = \int_0^\lambda d\lambda' \delta \hat{\Gamma}^n = - \left[ 2\varphi + \Psi + 2C^\alpha n^\beta + 2H\chi n^\alpha - \frac{d}{d\lambda} G^\alpha \right]_\lambda 
- \int_0^{\hat{\nu}_s} d\hat{r} \left[ (\alpha - \varphi) - \Psi - C_{\parallel} n^\beta - C_{\beta\gamma} n^\beta n^\gamma \right], \tag{6.88}
$$

where the quantities in the square bracket are evaluated at the source and the observer positions parametrized by $\lambda$ and $\lambda_o$. Note that the derivative $d\lambda$ along the photon path was considered only at the background level and we replaced it with the integration over the comoving distance $d\hat{r}$, all of which are valid only when the integrands are at the linear order in perturbations. The perturbations $(\delta \nu, \delta n^\alpha)_{\o}$ at the observer position are fixed in Eq. (6.80) as the boundary condition.

Before we proceed to obtain the source position $x^\nu_{\o}$ by integrating the geodesic equations once more over the affine parameter, we derive the expression for the observed redshift. The light emitted in the rest-frame of the source travels across the Universe, and its wavelength is stretched due to the expansion. With reference to the rest-frame wavelength or the emission frequency $\omega_s$ in the source rest-frame, the observed redshift $z$ is constructed by using the observed frequency $\omega_o$ at the observer as

$$
1 + z := \frac{\omega_s}{\omega_o} = \frac{k^a_0}{k^a_o} = \frac{(u^a_0 k^a_{\mu})_s}{(u^a_0 k^a_{\mu})_o} = \frac{a_o}{a_s} \left( 1 + \Delta \nu_s - \Delta \nu_o \right) := 1 + \frac{\delta z}{a_s}, \tag{6.89}
$$

where we used Eq. (6.76) and we defined the perturbation $\delta z$ in the observed redshift. In addition to the cosmic expansion, the perturbations along the light propagation affect the observed redshift, and the perturbation $\delta z$ in the observed redshift captures such effects of inhomogeneities. Noting that the observer time-coordinate is $\eta_o = \eta_o + \delta \eta_o$ and the expression for the perturbation $\delta \nu$ is Eq. (6.87), we can derive the expression for $\delta z$ as

$$
\delta z \equiv \mathcal{H}_o \delta \eta_o + \Delta \nu_s - \Delta \nu_o = - H\chi + (\mathcal{H} \delta \eta + H\chi)_o + [V_{\parallel} - \alpha + \Psi']_{\lambda_o} - \int_0^{\hat{\nu}_s} d\hat{r} (\alpha - \varphi - \Psi - C_{\parallel})'. \tag{6.90}
$$

The photon wavelength (hence the observed redshift) is affected by the peculiar velocity and the gravitational redshift of the observer and the source positions. The latter is called the Sachs-Wolfe effect. Furthermore, the time evolution of the gravitational potential along the line-of-sight also gives rise to the integrated Sachs-Wolfe effect.

### 6.4.4 Source Position in FRW Coordinates

The photon path is a straight line in a homogeneous universe, and the inhomogeneities in the real universe deflect the photon path from a straight line. We will begin the calculations by considering a homogeneous universe first and thereby obtaining the relation to the affine parameter set by our normalization condition in Eq. (6.73). Any position $x^\mu_{\lambda}$ along the photon path will be marked by the affine parameter $\lambda$, and we will use bar to indicate that the position is derived in a homogeneous universe:

$$
\bar{x}^\mu_{\lambda} - \bar{x}^\mu_o = \int_0^\lambda d\lambda' \bar{k}^\mu_{\lambda'} = (\lambda, -\lambda n^\alpha), \tag{6.91}
$$

where we used Eq. (6.87) and defined the perturbation $\delta z$ in the observed redshift.
where we set to zero the affine parameter at the observer $\lambda_o = 0$ and the position of the observer in a homogeneous universe is uniquely set $\bar{x}^\nu := (\bar{\eta}_o, 0)$. As a coordinate in the world-line manifold, the affine parameter is defined by the above equation as
\[
\lambda := \bar{\eta}_\lambda - \bar{\eta}_o = -\bar{r}_\lambda ,
\] (6.92)
hence the spatial position becomes
\[
\bar{x}^\alpha = -\lambda n^\alpha = \bar{r}_\lambda n^\alpha .
\] (6.93)

The observed redshift is the only way we can assign a physically meaningful distance to cosmological objects. Since the comoving distance $\bar{r}$ is often defined in terms of a redshift parameter $z$, we define the affine parameter $\lambda_z$ and the time coordinate $\bar{\eta}_z$ of the source in the background in terms of the observed redshift $z$ as
\[
\lambda_z := \bar{\eta}_z - \bar{\eta}_o = -\bar{r}_z ,
\]
\[
1 + z = \frac{a(\bar{\eta}_o)}{a(\bar{\eta}_z)} ,
\]
\[
\bar{r}_z = \int^z_0 d\bar{z}' \frac{dz'}{H(\bar{z}')} .
\] (6.94)

We have used bar for the position $\bar{x}^\mu = (\bar{\eta}, \bar{x}^\alpha)_\lambda$ along the photon path to indicate that this position is evaluated in a homogeneous universe, given the observed angle $n^\alpha$ in the observer rest-frame.

With $(\delta \nu, \delta n^\alpha)$ in Eqs. (6.87) and (6.88), we can derive the expression for any position $x^\mu_s$ along the path by integrating the perturbations over the affine parameter. First, we integrate the temporal part of the photon wavevector:
\[
\delta \eta_\lambda - \delta \eta_o = \lambda \left( \Delta \nu + \alpha_\chi - V_\parallel - \Psi_\parallel + H_\chi \right)_0 - \left( \frac{\lambda}{a} \right)_o + \int^{\bar{r}_s} \int^0_0 d\bar{r} \left( 2\alpha_\chi + 2H_\chi - \Psi_\parallel \right)
\]
\[
+ \int^{\bar{r}_s} \int^0_0 d\bar{r} \left( \alpha_\chi - \varphi_\chi - \Psi_\parallel - C_\parallel \right)^\prime
\] (6.95)
such that the time coordinate of the source position is $\eta_s = \bar{\eta}_s + \delta \eta_s$. However, since the distance is more physically related to the observed redshift, we first relate the affine parameter $\lambda_s$ at the source position to the observed redshift as
\[
\lambda_s = \bar{\eta}_s - \bar{\eta}_o \equiv \lambda_z + \Delta \lambda_s ,
\] (6.96)
where we defined the residual deviation $\Delta \lambda_s$ of the affine parameter $\lambda_s$ from $\lambda_z$. Then the time-coordinate of the source position becomes
\[
\eta_s = \bar{\eta}_s + \Delta \eta_s \equiv \bar{\eta}_z + \Delta \eta ,
\] (6.97)
and the definition of the perturbation $\delta z$ in Eq. (6.89) yields
\[
\Delta \eta_s = \frac{\delta z}{\dot{H}} .
\] (6.98)

Next, we integrate the spatial part of the photon wavevector to obtain the source position in a FRW coordinate. As mentioned, it proves convenient to express the source position $x^\mu_s$ around the position $\bar{x}^\mu$ inferred from the observed redshift and angle. Having computed the time distortion $\Delta \eta_s$ in Eq. (6.98), we will compute the spatial distortion $\Delta x^\alpha_s$ of the source position as
\[
x^\alpha_s := \bar{x}^\alpha + \Delta x^\alpha_s = \bar{x}^\alpha + \delta x^\alpha_0 - \lambda_s n^\alpha - \int^{\bar{r}_s} d\bar{r} \delta n^\alpha = \bar{x}^\alpha + \delta x^\alpha_0 - \Delta \lambda_s n^\alpha + \int^{\bar{r}_s} d\bar{r} \delta n^\alpha .
\] (6.99)

Given the privileged direction $n^\alpha$, we decompose the spatial distortion into the radial $\delta r$ distortion and the transverse distortion $\delta x^\alpha_\perp$ as
\[
\Delta x^\alpha_s \equiv \delta r \ n^\alpha + \Delta x^\alpha_\perp ,
\]
\[
\delta r = n_\alpha \Delta x^\alpha_s ,
\]
\[
0 = n_\alpha \Delta x^\alpha_\perp ,
\] (6.100)
where the radial distortion is
\[
\delta r = n_\alpha \delta x^\alpha_o - \lambda_s - \left[ \frac{\chi}{a} + G_\parallel \right]_0 \int^{\bar{r}_s} d\bar{r} \left( \alpha_\chi - \varphi_\chi - \Psi_\parallel - C_\parallel \right)
\]
\[
= (\chi_o + \delta \eta_o) - \frac{\delta z}{\dot{H}_z} + \int^{\bar{r}_s} d\bar{r} \left( \alpha_\chi - \varphi_\chi - \Psi_\parallel - C_\parallel \right) + n_\alpha (\delta x^\alpha_o + G^\alpha_o)_o - n_\alpha G^\alpha_s ,
\] (6.101)
where we used the null condition in Eq. (6.86) and the distortion in the time coordinate in Eq. (6.98). The expression for $\delta r$ is arranged in terms of gauge-invariant variables, isolating the gauge-dependent term $n_\alpha G^\alpha_s$, such that $\delta r = \delta r + n_\alpha \mathcal{L}^\alpha_s$. 

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Being the spatial distortion, the radial distortion $\delta r$ depends on the spatial position $x^\alpha_0$ of the observer and hence the coordinate shift $\delta x^\alpha_\nu$, but again it is independent of the rotation $\Omega^\alpha_\nu$ of the spatial tetrad vectors (same for the distortion $\Delta \eta_s$ in the time coordinate).

For the transverse components $\Delta x^\alpha_\perp$ of the spatial distortions, we have to integrate $\delta n^\alpha_{\perp}$ in Eq. (6.88) over the affine parameter as

$$
\int_0^{\bar{r}_z} d\bar{r} \delta n^\alpha_{\perp} = \bar{r}_z \left[ \delta n^\alpha + 2H^\alpha \bar{n} - \frac{d}{d\bar{x}} G^\alpha + 2\varphi^\alpha + \Psi^\alpha + 2C^\alpha_{\perp} n^\nu \right] - G^\alpha_{s} + G^\alpha_{o} \tag{6.102}
$$

$$
- \int_0^{\bar{r}_z} d\bar{r} \left[ 2H^\alpha \bar{n} + 2\varphi^\alpha + \Psi^\alpha + 2C^\alpha_{\perp} n^\nu \right] - \int_0^{\bar{r}_z} d\bar{r} \left( \bar{r}_z - \bar{r} \right) \left[ (\alpha - \varphi^\alpha)^\alpha - \Psi^\beta n^\nu - C^\beta_{\gamma\alpha} n^\nu n^\gamma \right] .
$$

Further splitting the transverse distortion by using $\theta^\alpha$ and $\phi^\alpha$

$$
\Delta x^\alpha_\perp \equiv \bar{r}_z (\delta \theta, \sin \theta \delta \phi) , \quad \bar{r}_z \delta \theta = \theta_s \Delta x^\alpha_\perp , \quad \bar{r}_z \sin \theta \delta \phi = \phi_s \Delta x^\alpha_\perp , \tag{6.103}
$$

we derive the angular distortions as

$$
\bar{r}_z \delta \theta = \bar{r}_z \theta_s \left[ -V^\alpha + C^\alpha_{\beta} n^\nu - \epsilon^\alpha_{ij} n^i \Omega^j \right] - \int_0^{\bar{r}_z} d\bar{r} \theta_s \left[ \Psi^\alpha + 2C^\alpha_{\perp} n^\nu \right] \tag{6.104}
$$

$$
- \int_0^{\bar{r}_z} d\bar{r} \left( \bar{r}_z - \bar{r} \right) \left[ (\alpha - \varphi^\alpha)^\alpha - \Psi^\beta n^\nu - C^\beta_{\gamma\alpha} n^\nu n^\gamma \right] + \theta_s \left( \delta x^\alpha + G^\alpha \right)_o - G^\alpha_{s} ,
$$

where we used $\delta n^\alpha_{\perp}$ in Eq. (6.88). For the azimuthal distortion $\sin \theta \delta \phi$, the above equation can be used with $\theta^\alpha$ replaced with $\phi^\alpha$. The derivative in the second integration in Eq. (6.104) cannot be pulled out for the vector and the tensor perturbations, because it is the derivative with respect to the observed angle due to $\theta_s$:

$$
\Psi^\beta_{\gamma\alpha} \theta^\alpha n^\nu = 1 \left( \frac{\partial}{\partial \theta} \Psi^\beta_{\nu \gamma} / \theta^\alpha \right) , \quad C^\beta_{\gamma\alpha} \theta^\alpha n^\nu n^\gamma = 1 \left( \frac{\partial}{\partial \theta} (C^\beta_{\nu \gamma} / \theta^\alpha) \right) . \tag{6.105}
$$

With such expressions, we can further simplify the angular distortions as

$$
\delta \theta = \theta_s \left[ -V^\alpha + C^\alpha_{\beta} n^\nu - \epsilon^\alpha_{ij} n^i \Omega^j \right] - \int_0^{\bar{r}_z} d\bar{r} \left( \bar{r}_z - \bar{r} \right) \left[ (\alpha - \varphi^\alpha)^\alpha - \Psi^\beta n^\nu - C^\beta_{\gamma\alpha} n^\nu n^\gamma \right] + \theta_s \left( \delta x^\alpha + G^\alpha \right)_o + G^\alpha_{s} , \tag{6.106}
$$

This equation should be compared to Eq. (6.11) in the standard lensing formalism. We readily notice the incompleteness of the formula in the standard formalism.

In summary, the source position $x^\mu_\nu$, given the observed redshift $z$ and the observed angle $n^i$, is expressed as the sum of the position $\tilde{x}^\mu_\nu$ inferred from these observables and the deviation $\Delta x^\mu_\nu$ around it:

$$
x^\mu_\nu (z, \theta, \phi) = (\bar{r}_z + \Delta \eta_s, \bar{r}_z + \delta r, \theta + \delta \theta, \phi + \delta \phi) = \tilde{x}^\mu_\nu + \Delta x^\mu_\nu , \quad \tilde{x}^\mu_\nu = (\bar{r}_z, \bar{r}_z n^i) , \tag{6.107}
$$

where the components of the source position is written in a spherical coordinate. These angular distortions of the source position replace the lensing potential in the lens equation as

$$
\tilde{s} = \tilde{n} + (\delta \theta, \delta \phi) . \tag{6.108}
$$

Given the angular distortion $(\delta \theta, \delta \phi)$ in Eq. (6.104), it is straightforward to compute the lensing convergence:

$$
-2\kappa \equiv -2 \left( 1 - \frac{1}{2} T_Y \right) \left( \cot \theta + \frac{\partial}{\partial \theta} \right) \delta \theta + \frac{\partial}{\partial \phi} \delta \phi \tag{6.109}
$$

$$
= (2V^\alpha - 3C^\alpha_{\nu \gamma})_o + \int_0^{\bar{r}_z} d\bar{r} \left( \frac{2n^\alpha - \tilde{\nabla}^\alpha}{\bar{r}_z} \right) \left( \Psi^\alpha + 2C^\alpha_{\perp} n^\nu \right) - \frac{2n^\alpha G^\alpha_{s} + 2n^\alpha G^\alpha_{o}}{\bar{r}_z} \tag{6.110}
$$

$$
- \int_0^{\bar{r}_z} d\bar{r} \left( \frac{\bar{r}_z - \bar{r}}{\bar{r}_z \bar{r}} \right) \left( \bar{r}_z - \bar{r} \right) \left( \varphi^\alpha - \varphi^\alpha - \Psi^\beta - C^\beta_\gamma \right) - \frac{1}{\bar{r}_z} \tilde{\nabla}^\alpha G^\alpha_{s} ,
$$

where all the terms are arranged in terms of gauge-invariant variables except two terms multiplied with $G^\alpha_{\nu}$, such that the gravitational lensing convergence gauge-transforms as

$$
\tilde{\kappa} = \kappa + \frac{n^\alpha \tilde{L}^\alpha}{\bar{r}_z} - \frac{1}{2\bar{r}_z} \tilde{\nabla}^\alpha L^\alpha . \tag{6.111}
$$
Bibliography


