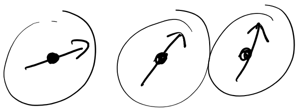
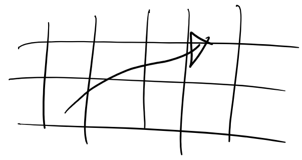
SPH

Lagrangian Method



Eulerian



Physics : Equation of Motion

$\underline{F} = m\underline{a}$

 δV small "parcel" of the fluid

$\underline{F} = \rho \delta V \frac{d\underline{u}}{dt}$ fluid velocity with respect to the moving frame of reference.

$\frac{d\underline{u}}{dt}$ = "Time derivative following the motion" $\equiv \frac{\underline{u}(\underline{x}+\underline{u}\delta t, t+\delta t) - \underline{u}(\underline{x},t)}{\delta t}$

$$\frac{dF}{dt} = \frac{1}{\cancel{\delta t}}\Big\{\underbrace{F(\underline{x}+\underline{u}\delta t, t+\delta t)}_{\text{Taylor expand}} - F(\underline{x},t)\Big\}$$

$$\Big\{\cancel{F(\underline{x},t)} + \frac{\partial F}{\partial \underline{x}}\underline{u}\cancel{\delta t} + \frac{\partial F}{\partial t}\cancel{\delta t} - \cancel{F(\underline{x},t)}\Big\}$$

$$\boxed{\frac{dF}{dt} = (\underline{u}\cdot\nabla)F + \frac{\partial F}{\partial t}}$$

Convective derivative

$$\frac{d\underline{u}}{dt} = \underline{u}\cdot\nabla\underline{u} + \frac{\partial \underline{u}}{\partial t}$$

$\rho\delta V\frac{d\underline{u}}{dt} = \rho\delta V g$ No. We need to include the contribution of pressure

$$\frac{d\underline{u}}{dt} = g - \frac{\nabla P}{\rho}$$

Can do the derivation later

What variables do we have, $\underbrace{\underline{u}, P \text{ and } \rho}_{\text{5 variables}}$

We have only 3 equations of motion in $\frac{d\underline{u}}{dt}$ ...

1. Continuity equation : conservation of mass.
2. Energy equation : ✓

Continuity Eq.

1. $$\frac{\partial \rho}{\partial t} + \nabla \cdot \underbrace{(\rho \underline{u})}_{\underline{u} \cdot \nabla \rho + \rho \nabla \cdot \underline{u}} = 0$$ Continuity Eq.

$$\boxed{\frac{d\rho}{dt} + \rho \nabla \cdot \underline{u} = 0}$$

Lagrangian Continuity Equation.

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2. Conservation of Energy

$$\boxed{\frac{de}{dt} = -\left(\frac{P}{\rho}\right) \nabla \cdot \underline{u}}$$

e : Specific Internal energy

per unit mass

Total energy is given by:

$$E = \rho \cdot \left(\frac{1}{2} \underline{u} \cdot \underline{u} + e\right)$$

What variables? $\underline{u}, \rho, e, P$ — 6 variables and 5 equation

P, e, ρ $\longrightarrow$ Equation of State (EOS)

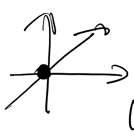
$PV = nRT$ Ideal Gas EOS

$$\boxed{e = \frac{P}{\rho(\gamma - 1)}}$$ Ideal Gas EOS

$$\gamma = \frac{f+2}{f}$$

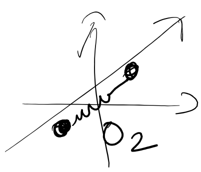
f is the number of degrees of freedom

Monoatomic



$f = 3$ $\quad \gamma = \frac{5}{3}$

He, ionized hydrogen



O_2 $\quad$ 6

$$e = \frac{k_B T}{\mu \, m_H}$$

k_B: Bolzmann constant

m_H: mass of Hydrogen

e^- p^+

$\mu = 0.5$

$\sim \mu m_H$ — mass of Hydrogen

μ — mean molar weight

SPH - use particles to follow the flow and consider various integrals in the calculation of the conservation laws.

$$\underbrace{A_I(\underline{r})}_{\text{interpolant}} = \int_{\text{all space}} A(\underline{r}')\, W(\underline{r}-\underline{r}';h)\, d\underline{r}'$$

"kernel"

1. $\int W \, d\underline{r} = 1$

2. $\lim_{h \to 0} W(\underline{r}-\underline{r}';h) = \delta(\underline{r}-\underline{r}')$

limit of a gaussian

approximate via a sum

$$A_S(\underline{r}) = \sum_b m_b \frac{A_b}{\rho_b} W(\underline{r}-\underline{r}_b;h)$$

$$* \quad \boxed{\rho_S(\underline{r}) = \sum_b m_b W(\underline{r}-\underline{r}_b;h)}$$

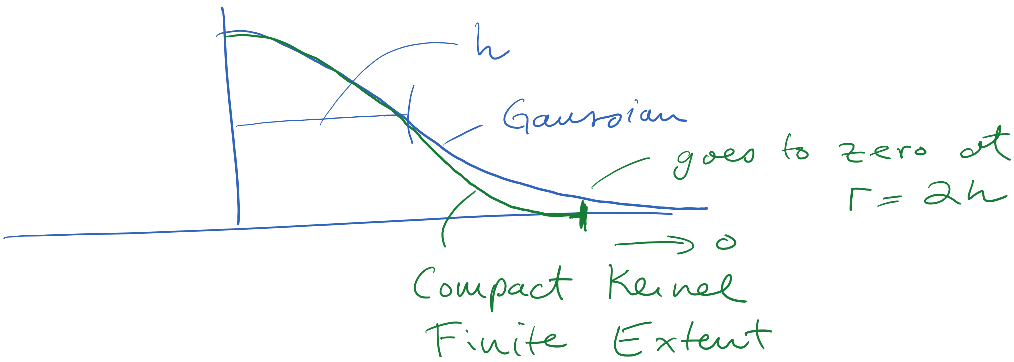
We also need to approximate (or interpolate) gradients of functions:

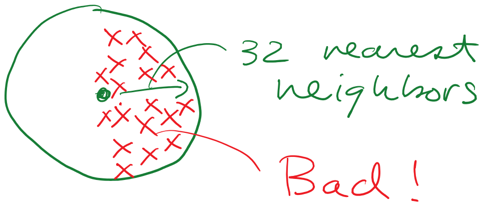
$$\nabla A_S(\underline{r}) = \sum_b m_b \frac{A_b}{\rho_b} \underbrace{\nabla W(\underline{r}-\underline{r}_b;h)}_{\text{can calculate in advance}}$$

Instead (higher accuracy) we use:

$$\rho \nabla A = \nabla(\rho A) - A \nabla \rho$$

Kernels: Gaussian $W(x,h) = \frac{1}{h\sqrt{\pi}} e^{-\left(\frac{x^2}{h^2}\right)}$





Monahan Kernel: "cubic spline kernel"

$$W(r; h) = \frac{\sigma}{h^d} \begin{cases} 6\left(\frac{r}{h}\right)^3 - 6\left(\frac{r}{h}\right)^2 + 1 \,, & 0 \leq \frac{r}{h} < \frac{1}{2} \\ 2\left(1 - \left(\frac{r}{h}\right)\right)^3 \,, & \frac{1}{2} \leq \frac{r}{h} \leq 1 \\ 0 \,, & \frac{r}{h} > 1 \end{cases}$$

d is dimension

Normalizing Constant $\sigma = \begin{cases} 4/3 \text{ in 1-D} & d=1 \\ 40/7\pi \text{ in 2-D} \\ 8/\pi \text{ in 3-D} \end{cases}$

∇W?