Computational Astrophysics 4

The Godunov method

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Outline

- Hyperbolic system of conservation laws
- Finite difference approximation
- The Modified Equation
- The Upwind scheme
- Von Neumann Analysis
- The Godunov Method
- Riemann solvers
- 2D Godunov schemes

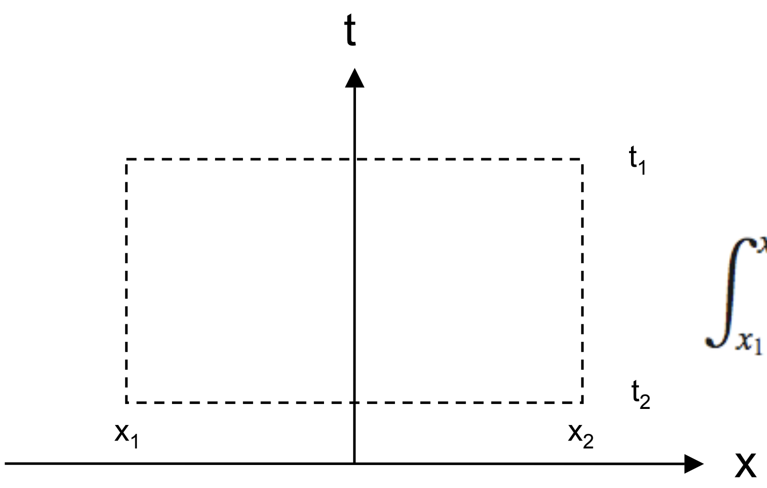
HS of CL

System of conservation laws

$$\partial_t \mathbf{U} + \partial_x \mathbf{F} = 0$$

- Vector of conservative variables $\quad {}^T\mathbf{U} = (\rho, \rho u, E)$

- Flux function $\quad {}^T\mathbf{F} = (\rho u, \rho u^2 + P, (E + P)u)$



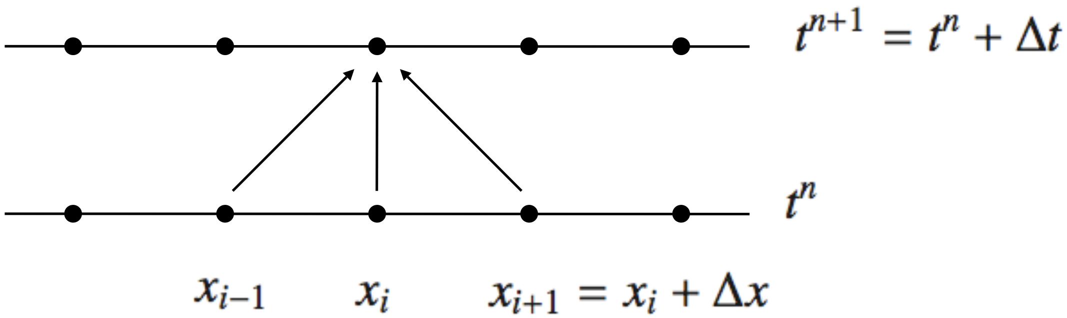
Integral form

$$\int_{x_1}^{x_2} \int_{t_1}^{t_2} \mathrm{dxdt}\, (\partial_t \mathbf{U} + \partial_x \mathbf{F}) = 0$$

$$\int_{x_1}^{x_2} \mathrm{dx}\, (\mathbf{U}(t_2) - \mathbf{U}(t_1)) + \int_{t_1}^{t_2} \mathrm{dt}\, (\mathbf{F}(x_2) - \mathbf{F}(x_1)) = 0$$

$$\mathcal{U}(t_2) - \mathcal{U}(t_1) + \mathcal{F}(x_2) - \mathcal{F}(x_1) = 0$$

Finite difference scheme



$$u_i^n = u(x_i, t^n) \qquad \partial_x u \simeq \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} \qquad \partial_t u \simeq \frac{u_i^{n+1} - u_i^n}{\Delta t}$$

Finite difference approximation of the advection equation

$$\partial_t u + a \partial_x u = 0 \quad \Longrightarrow \quad \frac{u_i^{n+1} - u_i^n}{\Delta t} + a \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} = 0$$

The Modified Equation

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}+a\frac{u_{i+1}^n-u_{i-1}^n}{2\Delta x}=0$$

Taylor expansion in time up to second order

$$u_i^{n+1}=u_i^n+\Delta t\left(\frac{\partial u}{\partial t}\right)+\frac{(\Delta t)^2}{2}\left(\frac{\partial^2 u}{\partial t^2}\right)$$

Taylor expansion in space up to second order

$$u_{i+1}^n=u_i^n+\Delta x\left(\frac{\partial u}{\partial x}\right)+\frac{(\Delta x)^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)$$

$$u_{i-1}^n=u_i^n-\Delta x\left(\frac{\partial u}{\partial x}\right)+\frac{(\Delta x)^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)$$

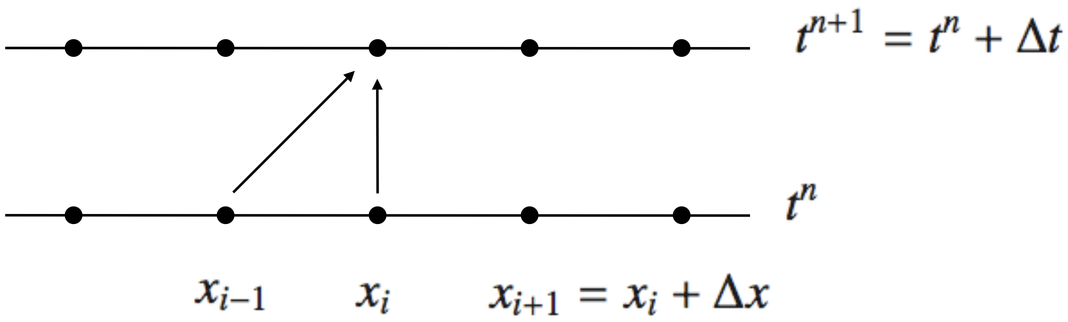
The advection equation becomes the advection-diffusion equation

$$\left(\frac{\partial u}{\partial t}\right)+a\left(\frac{\partial u}{\partial x}\right)=-\frac{\Delta t}{2}\left(\frac{\partial^2 u}{\partial t^2}\right)+O(\Delta t^2,\Delta x^2)$$

$$\left(\frac{\partial u}{\partial t}\right)+a\left(\frac{\partial u}{\partial x}\right)=-a^2\frac{\Delta t}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)+O(\Delta t^2,\Delta x^2)$$

Negative diffusion coefficient: the scheme is *unconditionally unstable*

The Upwind scheme



a>0: use only upwind values, discard downwind variables

$$\partial_x u \simeq \frac{u_i^n - u_{i-1}^n}{\Delta x} \quad \Rightarrow \quad \frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_i^n - u_{i-1}^n}{\Delta x} = 0$$

Taylor expansion up to second order:

$$\left(\frac{\partial u}{\partial t}\right) + a\left(\frac{\partial u}{\partial x}\right) = -\frac{\Delta t}{2}\left(\frac{\partial^2 u}{\partial t^2}\right) + a\frac{\Delta x}{2}\left(\frac{\partial^2 u}{\partial x^2}\right) + O(\Delta t^2, \Delta x^2)$$

Upwind scheme is stable if C<1, with $C = a\dfrac{\Delta t}{\Delta x}$

$$\left(\frac{\partial u}{\partial t}\right) + a\left(\frac{\partial u}{\partial x}\right) = a\frac{\Delta x}{2}(1 - C)\left(\frac{\partial^2 u}{\partial x^2}\right) + O(\Delta t^2, \Delta x^2)$$

Von Neumann analysis

Fourier transform the current solution: $u_i^n = \sum_k A_k^n \exp(-ikx_i)$

Evaluate the amplification factor of the 2 schemes.

Fromm scheme: $u_i^{n+1} = u_i^n - \frac{C}{2}u_{i+1}^n + \frac{C}{2}u_{i-1}^n$

$$A_k^{n+1} = A_k^n \left(1 - \frac{C}{2}\exp(-ik\Delta x) + \frac{C}{2}\exp(ik\Delta x)\right)$$

$$\omega^2 = \frac{|A_k^{n+1}|^2}{|A_k^n|^2} = 1 + C^2 \sin(k\Delta x)^2$$

$\omega>1$: the scheme is unconditionally unstable

Upwind scheme: $u_i^{n+1} = u_i^n(1-C) + Cu_{i-1}^n$

$$A_k^{n+1} = A_k^n\left(1 - C + C\exp(ik\Delta x)\right)$$

$$\omega^2 = \frac{|A_k^{n+1}|^2}{|A_k^n|^2} = 1 - 2C(1-C)(1-\cos(k\Delta x))$$

$\omega<1$ if $C<1$: the scheme is stable under the Courant condition.

The advection-diffusion equation

Finite difference approximation of the advection equation:

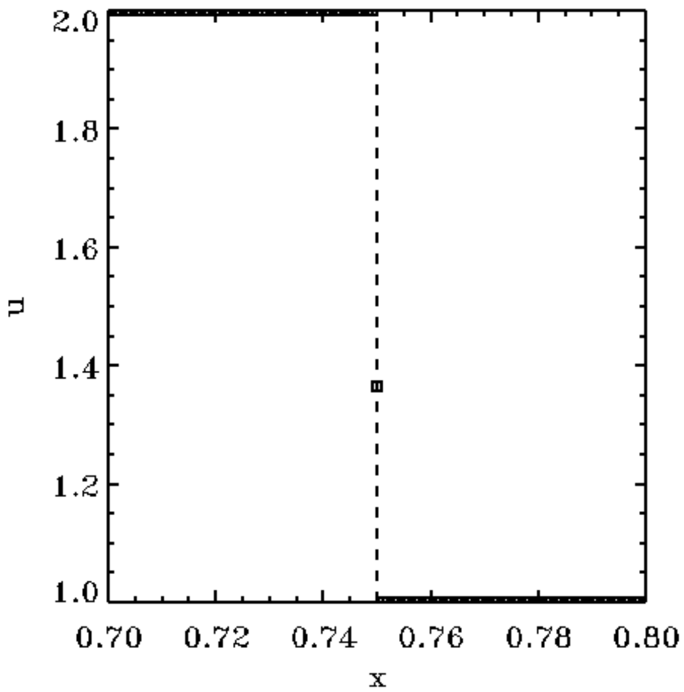
$$\left(\frac{\partial u}{\partial t}\right) + a\left(\frac{\partial u}{\partial x}\right) = \eta\left(\frac{\partial^2 u}{\partial x^2}\right)$$

Central differencing unstable: $\eta < 0$

Upwind differencing is stable: $\eta > 0$ $\quad \eta = a\frac{\Delta x}{2}(1 - C)$

Smearing of initial discontinuity:

"numerical diffusion"



Thickness increases as $\sqrt{\eta t}$

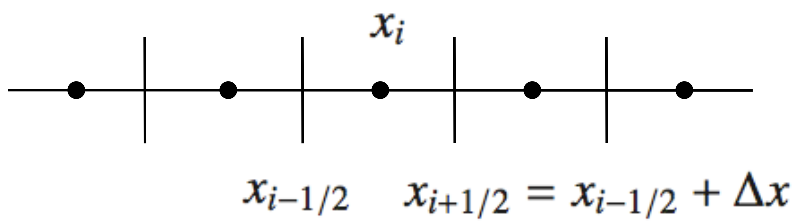
The Godunov method

Sergei Konstantinovich Godunov

Sergei Konstantinovich Godunov

Born	17th July, 1929 Moscow

Finite volume scheme



Finite volume approximation of the advection equation:

$$u_i^n = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} u(x, t^n)\mathrm{dx}$$

Use integral form of the conservation law:

$$\int_{x_{i-1/2}}^{x_{i+1/2}}\int_{t^n}^{t^{n+1}} \mathrm{dxdt}\,(\partial_t u + a\partial_x u) = 0$$

Exact evolution of volume averaged quantities:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\frac{u_{i+1/2}^{n+1/2} - u_{i-1/2}^{n+1/2}}{\Delta x} = 0$$

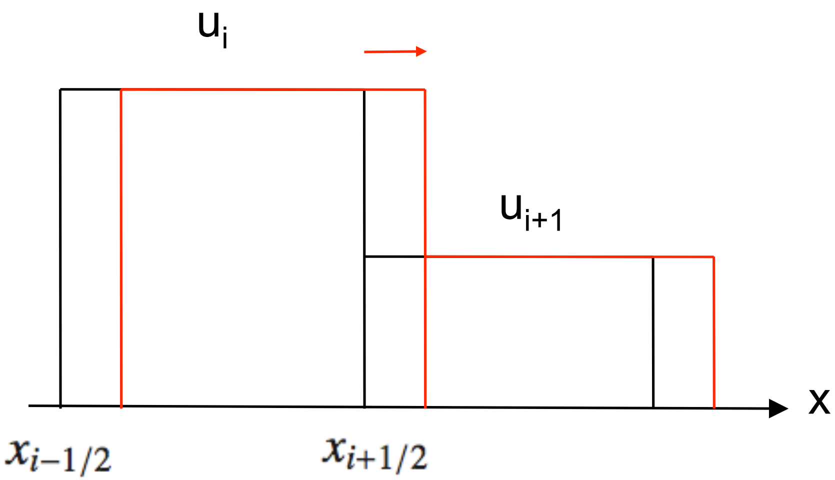
Time averaged flux function: $$u_{i+1/2}^{n+1/2} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} u(x_{i+1/2}, t)\mathrm{dt}$$

Godunov scheme for the advection equation

The time averaged flux function: $u_{i+1/2}^{n+1/2} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} u(x_{i+1/2}, t)\mathrm{dt}$

is computed using the solution of the Riemann problem defined

at cell interfaces with piecewise constant initial data.



For all t>0:

$$u(x_{i+1/2}, t) = u_i^n \quad \text{if} \quad a > 0$$
$$u(x_{i+1/2}, t) = u_{i+1}^n \quad \text{if} \quad a < 0$$

The Godunov scheme for the advection equation is identical to the upwind finite difference scheme.

Godunov scheme for hyperbolic systems

The system of conservation laws

$$\partial_t \mathbf{U} + \partial_x \mathbf{F} = 0$$

is discretized using the following integral form:

$$\frac{\mathbf{U}_i^{n+1} - \mathbf{U}_i^n}{\Delta t} + \frac{\mathbf{F}_{i+1/2}^{n+1/2} - \mathbf{F}_{i-1/2}^{n+1/2}}{\Delta x} = 0$$

The time average flux function is computed using the self-similar solution of the inter-cell Riemann problem:

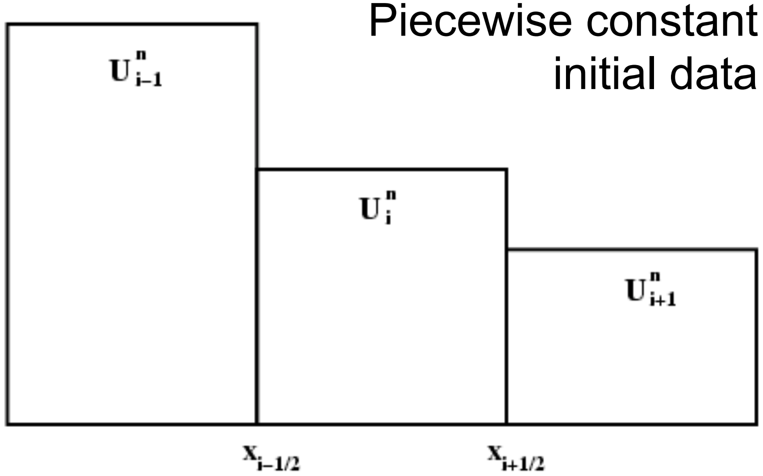
$$\mathbf{U}^*_{i+1/2}(x/t) = \mathcal{RP}\left[\mathbf{U}_i^n, \mathbf{U}_{i+1}^n\right]$$

$$\mathbf{F}_{i+1/2}^{n+1/2} = \mathbf{F}(\mathbf{U}^*_{i+1/2}(0))$$

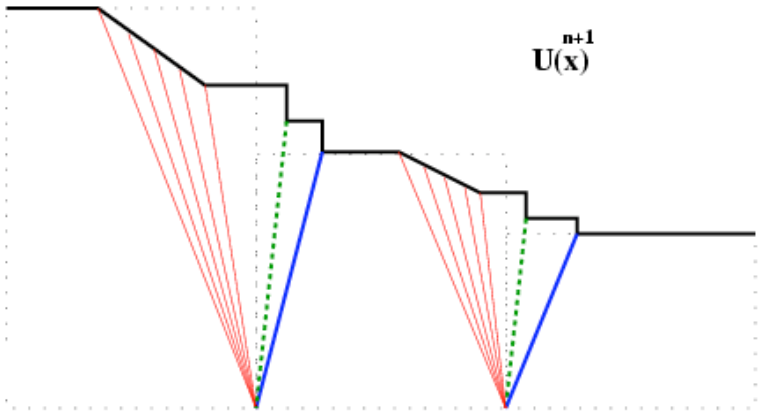
This defines the Godunov flux:

$$\mathbf{F}_{i+1/2}^{n+1/2} = \mathbf{F}^*(\mathbf{U}_i^n, \mathbf{U}_{i+1}^n)$$

Piecewise constant initial data



- **Godunov, S. K.** (1959), A Difference Scheme for Numerical Solution of Discontinuos Solution of Hydrodynamic Equations, *Math. Sbornik*, ***47**, 271-306*, translated US Joint Publ. Res. Service, JPRS 7226, 1969.



Advection: 1 wave, Euler: 3 waves, MHD: 7 waves

Higher Order Godunov schemes

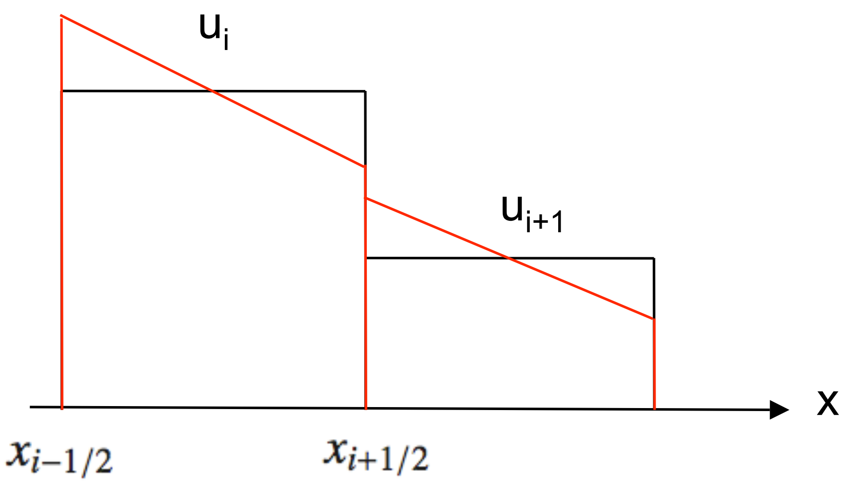
Godunov method is stable but very diffusive. It was abandoned for two decades, until…

Bram Van Leer

- **van Leer, B.** (1979), Towards the Ultimate Conservative Difference Scheme, V. A Second Order Sequel to Godunov's Method, *J. Com. Phys..*, *32*, 101–136.

Second Order Godunov scheme

Piecewise linear approximation of the solution:



The linear profile introduces a length scale: the Riemann solution is not self-similar anymore:

$$\mathbf{F}^{n+1/2}_{i+1/2} \neq \mathbf{F}(\mathbf{U}^*_{i+1/2}(0))$$

The flux function is approximated using a *predictor-corrector* scheme:

$$\mathbf{F}^{n+1/2}_{i+1/2} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \mathbf{F}(x_{i+1/2}, t)\,\mathrm{d}t \quad \Longrightarrow \quad \mathbf{F}^{n+1/2}_{i+1/2} \simeq \mathbf{F}(\mathbf{U}^*_{i+1/2}(\frac{\Delta t}{2}))$$

The *corrected* Riemann solver has now *predicted* states as initial data:

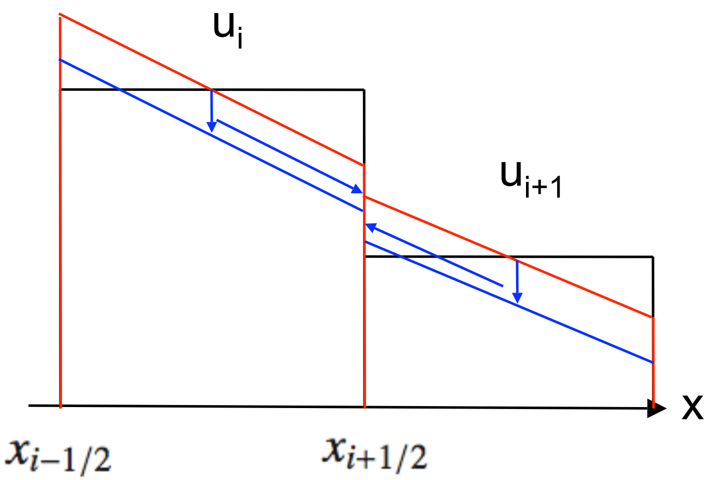
$$\mathbf{U}^*_{i+1/2}(x/t) = \mathcal{RP}\left[\mathbf{U}^{n+1/2}_{i+1/2,L}, \mathbf{U}^{n+1/2}_{i+1,R}\right]$$

Predictor Step for the advection equation

The predicted states are computed using a Taylor expansion in space and time:

$$u^{n+1/2}_{i+1/2,L} = u^n_i + \frac{\Delta t}{2}\left(\frac{\partial u}{\partial t}\right)_i + \frac{\Delta x}{2}\left(\frac{\partial u}{\partial x}\right)_i$$

$$u^{n+1/2}_{i+1/2,R} = u^n_{i+1} + \frac{\Delta t}{2}\left(\frac{\partial u}{\partial t}\right)_{i+1} - \frac{\Delta x}{2}\left(\frac{\partial u}{\partial x}\right)_{i+1}$$



Second order predicted states are the new initial conditions for the Riemann solver:

$$u^{n+1/2}_{i+1/2,L} = u^n_i + (1-C)\frac{\Delta x}{2}\left(\frac{\partial u}{\partial x}\right)_i \qquad u^{n+1/2}_{i+1/2,R} = u^n_{i+1} - (1+C)\frac{\Delta x}{2}\left(\frac{\partial u}{\partial x}\right)_{i+1}$$

The *corrected* flux function is the *upwind* predicted state:

$$f^{n+1/2}_{i+1/2} = a u^{n+1/2}_{i+1/2,L} \quad \text{if} \quad a > 0 \qquad f^{n+1/2}_{i+1/2} = a u^{n+1/2}_{i+1/2,R} \quad \text{if} \quad a < 0$$

Modified equation for the second order scheme

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}+a\frac{u_i^n-u_{i-1}^n}{\Delta x}+\frac{a}{2}(1-C)\left[\left(\frac{\partial u}{\partial x}\right)_i-\left(\frac{\partial u}{\partial x}\right)_{i-1}\right]=0$$

Taylor expansion in space and time up to third order:

$$u_i^{n+1}=u_i^n+\Delta t\left(\frac{\partial u}{\partial t}\right)+\frac{(\Delta t)^2}{2}\left(\frac{\partial^2 u}{\partial t^2}\right)+\frac{(\Delta t)^3}{6}\left(\frac{\partial^3 u}{\partial t^3}\right)$$

$$u_{i-1}^n=u_i^n-\Delta x\left(\frac{\partial u}{\partial x}\right)+\frac{(\Delta x)^2}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)-\frac{(\Delta x)^3}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)$$

$$\left(\frac{\partial u}{\partial x}\right)_{i-1}=\left(\frac{\partial u}{\partial x}\right)_i-\Delta x\left(\frac{\partial^2 u}{\partial x^2}\right)+\frac{(\Delta x)^2}{2}\left(\frac{\partial^3 u}{\partial x^3}\right)$$

We obtain a *dispersive term* as leading-order error.

Von Neumann analysis says the scheme is stable for C<1.

$$\left(\frac{\partial u}{\partial t}\right)+a\left(\frac{\partial u}{\partial x}\right)=a\frac{\Delta x^2}{6}(1-C)(\frac{1}{2}-C)\left(\frac{\partial^3 u}{\partial x^3}\right)+O(\Delta t^3,\Delta x^3)$$

Monotonicity preserving schemes

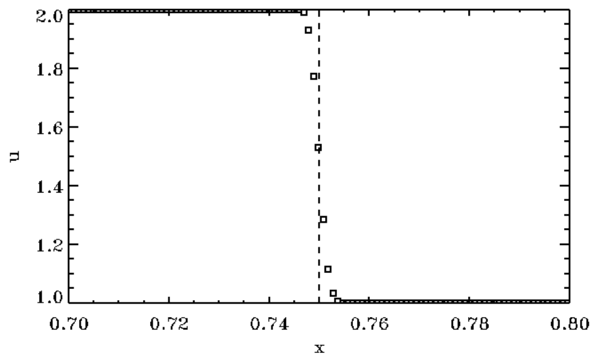
We use the central finite difference approximation for the slope:

$$\left(\frac{\partial u}{\partial x}\right)_i = \frac{\Delta u_i}{\Delta x} = \frac{u_{i+1} - u_{i-1}}{2}$$

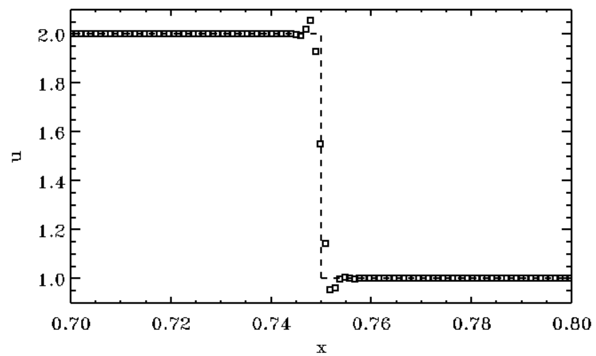
Second order linear scheme.

In this case, the solution is oscillatory, and therefore non physical.

first order



second order



Oscillations are due to the *non monotonicity* of the numerical scheme.

A scheme is monotonicity preserving if:

- No new local extrema are created in the solution
- Local minimum (maximum) non decreasing (increasing) function of time.

Godunov theorem: only first order linear schemes are monotonicity preserving !

Slope limiters

Harten introduced the Total Variation of the numerical solution:

$$TV^n = \sum_i^n |u_{i+1} - u_i|$$

Harten's theorem: a Total Variation Diminishing (TVD) scheme is monotonicity preserving.

$$TV^{n+1} \leq TV^n$$

Design non-linear TVD second order scheme using slope limiters:

$$\left(\frac{\partial u}{\partial x}\right)_i = \frac{\Delta u_i}{\Delta x} = \lim(u_{i-1}, u_i, u_{i+1}) \left(\frac{u_{i+1} - u_{i-1}}{2}\right)$$

where the slope limiter is a non-linear function satisfying:

$$0 \leq \lim(u_{i-1}, u_i, u_{i+1}) \leq 1$$

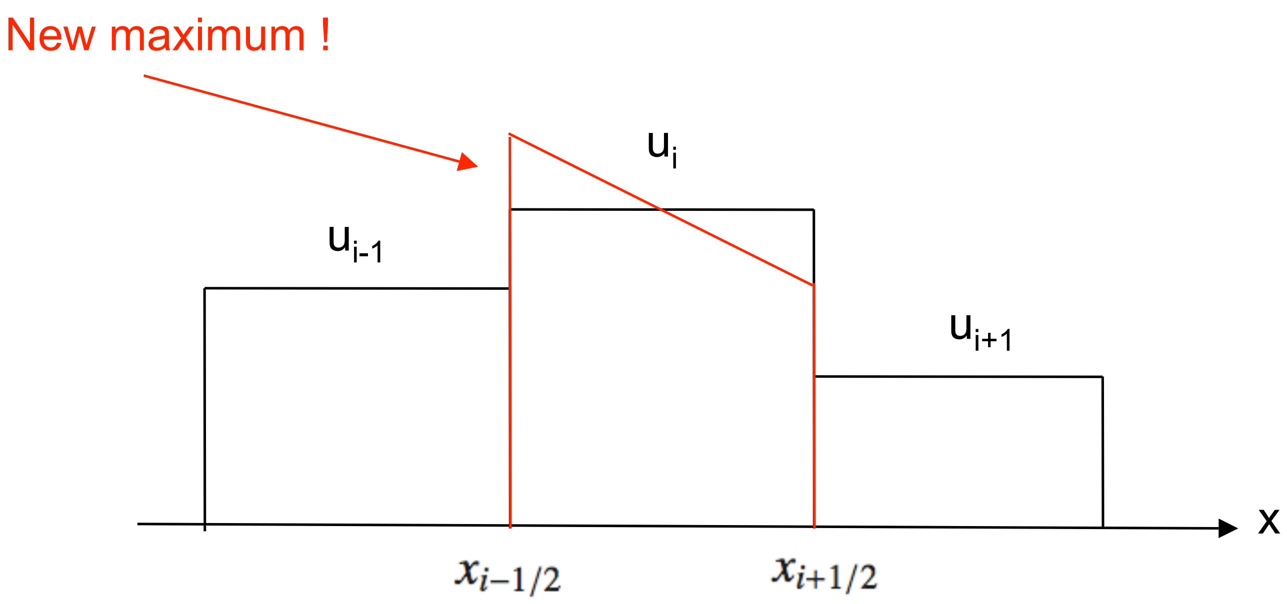
- Harten, Ami (1983), "High resolution schemes for hyperbolic conservation laws", *J. Comput. Phys* **49**: 357-393, doi:10.1006/jcph.1997.5713

No local extrema

$$\left(\frac{\partial u}{\partial x}\right)_i = \frac{\Delta u_i}{\Delta x} = \lim(u_{i-1}, u_i, u_{i+1})\left(\frac{u_{i+1} - u_{i-1}}{2}\right)$$

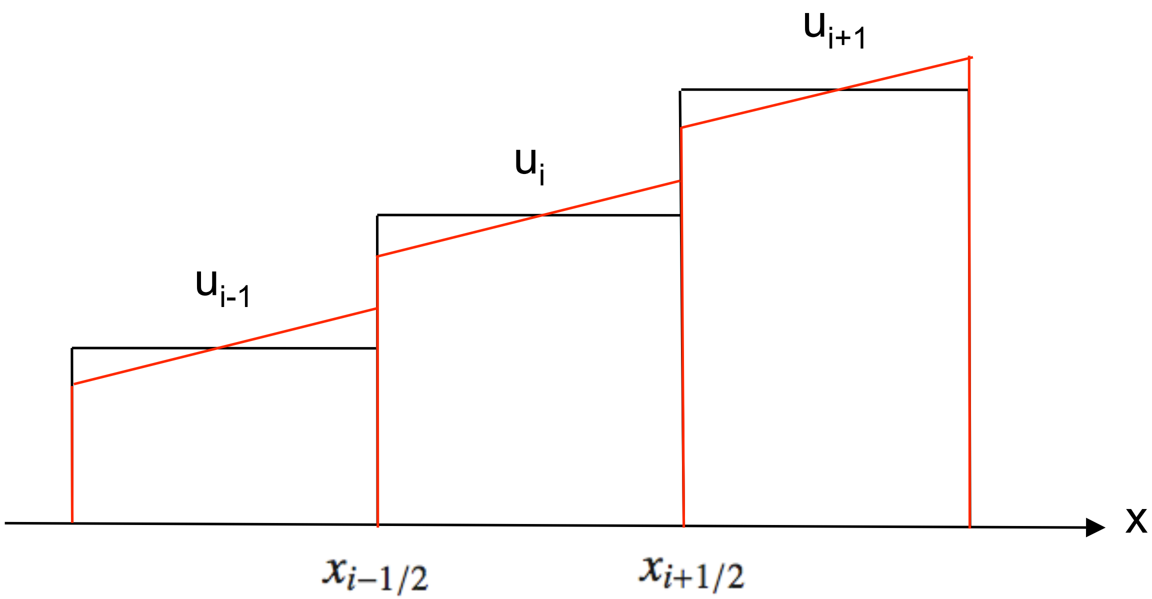
We define 3 local slopes: left, right and central slopes

$\Delta u_L = u_i - u_{i-1}$ $\Delta u_R = u_{i+1} - u_i$ and $\Delta u_C = \dfrac{u_{i+1} - u_{i-1}}{2}$



For all slope limiters: $\Delta u_i = 0 \quad \text{if} \quad \Delta u_L \Delta u_R < 0$

The *minmod* slope



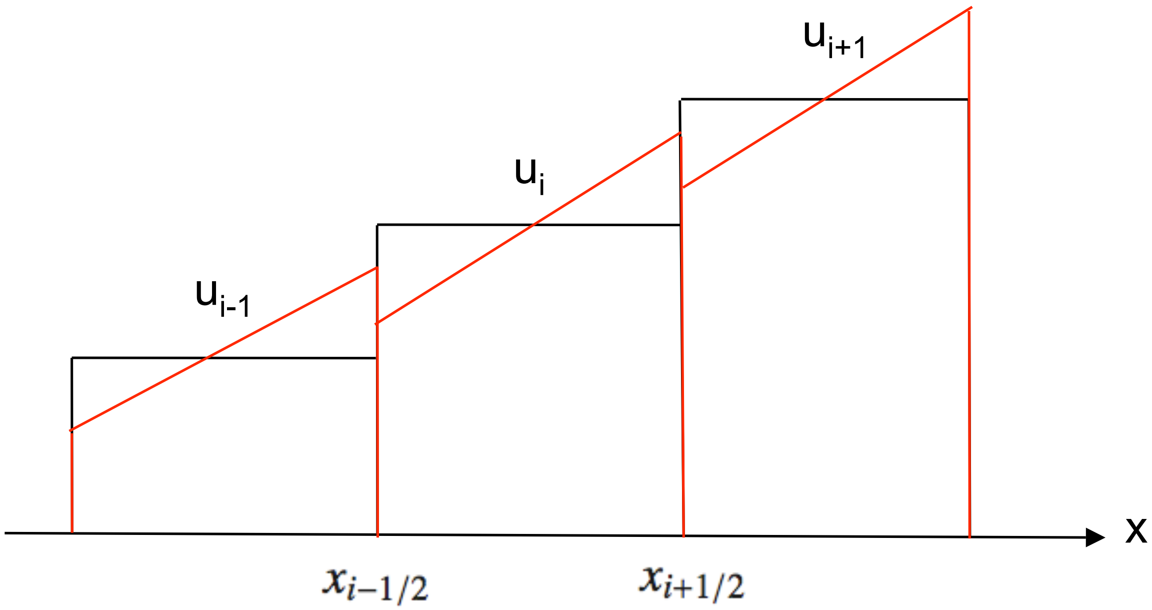
Linear reconstruction is monotone at time t^n

$$u^n_{i+1/2,L} = u^n_i + \frac{\Delta u_i}{2} \qquad u^n_{i-1/2,R} = u^n_i - \frac{\Delta u_i}{2}$$

Minmod slope limiting is never truly second order !

$$u^n_{i+1/2,L} \leq u^n_{i+1/2,R} \qquad \Delta u_i = \min(\Delta u_L, \Delta u_R)$$

The *moncen* slope



Extreme values must be bounded by the *initial average* states.

$$u^n_{i-1/2,R} = u^n_i - \frac{\Delta u_i}{2} \qquad u^n_{i+1/2,L} = u^n_i + \frac{\Delta u_i}{2}$$

$$u^n_{i-1} \le u^n_{i-1/2,R} \le u^n_i \qquad u^n_i \le u^n_{i+1/2,L} \le u^n_{i+1}$$

$$\Delta u_i = \min(2\Delta u_L, \Delta u_C, 2\Delta u_R)$$

The *superbee* slope

Predicted states must be bounded by the initial average states.

$$u_{i+1/2,L}^{n+1/2} = u_i^n + (1 - C)\frac{\Delta u_i}{2}$$

$$u_{i+1/2,R}^{n+1/2} = u_{i+1}^n - (1 + C)\frac{\Delta u_{i+1}}{2}$$

TVD constraint is preserved by the Riemann solver.

$$u_i^n \le u_{i+1/2,L}^{n+1/2} \le u_{i+1}^n$$

$$u_{i-1}^n \le u_{i-1/2,R}^{n+1/2} \le u_i^n$$

The Courant factor now enters the slope definition.

$$\Delta u_i = \min\left(\frac{2}{1 + C}\Delta u_L, \frac{2}{1 - C}\Delta u_R\right)$$

The ultrabee slope

Use the final state to compute the slope limiter.

$$u_i^{n+1} = u_i^n(1-C) + u_{i-1}^n C - \frac{C}{2}(1-C)\left(\Delta u_i - \Delta u_{i-1}\right) = 0$$

Upwind Total Variation constraint.

$$u_{i-1}^n \le u_i^{n+1} \le u_i^n$$

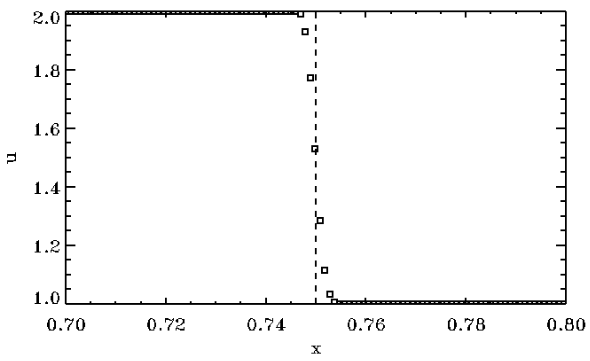
Strict Total Variation preserving limiter.

$$\text{if} \quad C > 0 \quad \Delta u_i = \min(\frac{2}{C}\Delta u_L, \frac{2}{1-C}\Delta u_R)$$

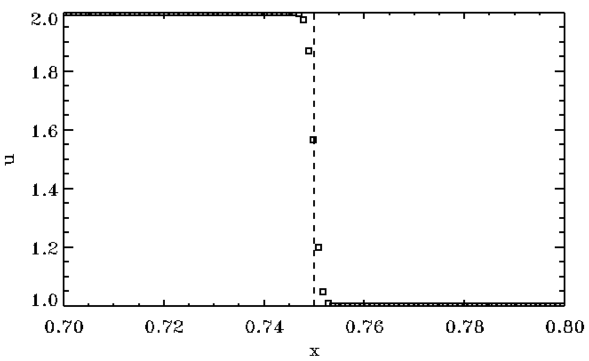
$$\text{if} \quad C < 0 \quad \Delta u_i = \min(\frac{2}{1+C}\Delta u_L, \frac{2}{-C}\Delta u_R)$$

Summary: slope limiters

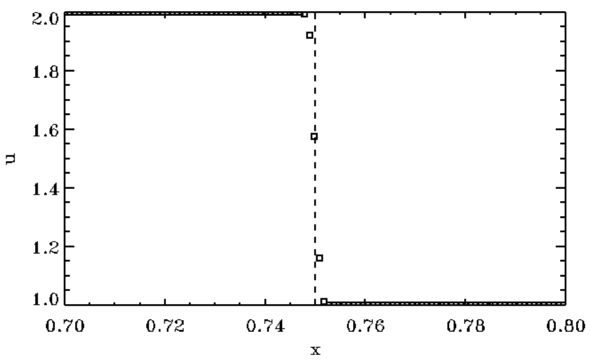
first order



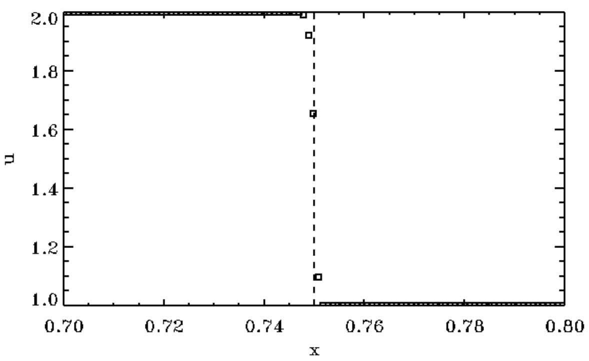
minmod



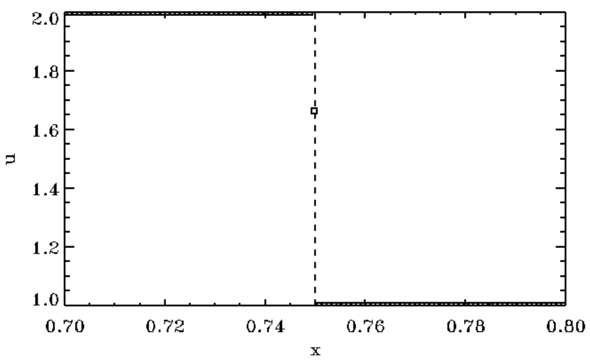
moncen



superbee



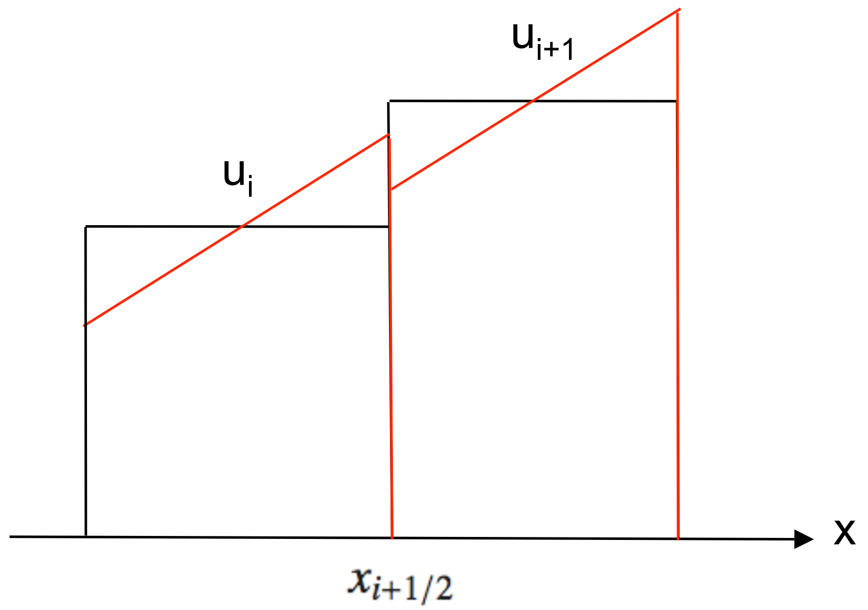
ultrabee



Summary: slope limiters

The previous analysis is valid only for the advection equation.

Non-linear systems: the wave speeds depend on the initial states (L and R).

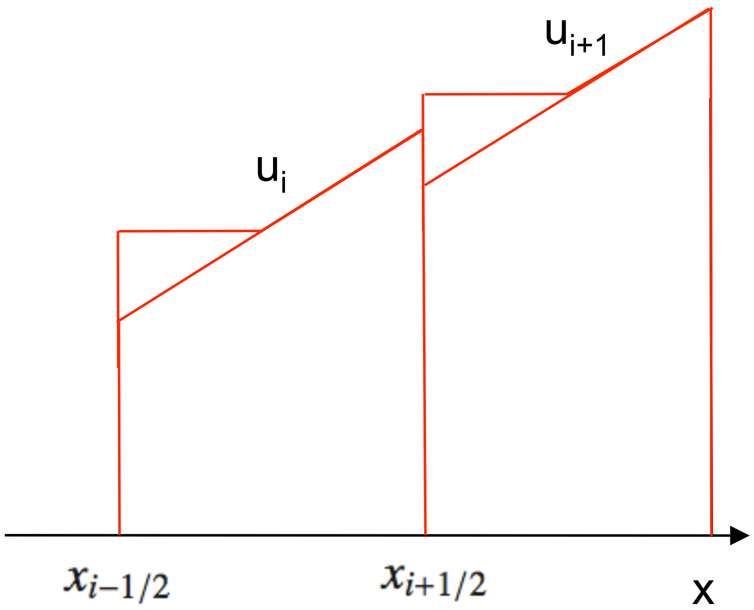


MinMod is the only monotone slope limiter before the Riemann solver !

Superbee and Ultrabee must not be used for non-linear systems !

MonCen can be used, but with care: the characteristics tracing method.

Non-linear systems: characteristics tracing.



Non-linear Riemann problems: waves speeds depend on the input states.

TVD schemes are not necessary monotone.

Modify the predictor step according to the local Riemann solution: Piecewise Linear Method (PLM) and Piecewise Parabolic Method (PPM).

If $(C_k)_i > 0 \quad (\alpha_k)^{n+1/2}_{i+1/2,L} = (\alpha_k)^n_i + (1 - (C_k)_i)\dfrac{(\Delta\alpha_k)_i}{2}$

else $(\alpha_k)^{n+1/2}_{i+1/2,L} = (\alpha_k)^n_i$

$$C_- = (u - a)\frac{\Delta t}{\Delta x}$$

$$C_0 = u\frac{\Delta t}{\Delta x}$$

$$C_+ = (u + a)\frac{\Delta t}{\Delta x}$$

If $(C_k)_{i+1} < 0 \quad (\alpha_k)^{n+1/2}_{i+1/2,R} = (\alpha_k)^n_{i+1} - (1 + (C_k)_{i+1})\dfrac{(\Delta\alpha_k)_i}{2}$

else $(\alpha_k)^{n+1/2}_{i+1/2,R} = (\alpha_k)^n_{i+1}$

- Colella, P. and Woodward, P., "The Piecewise parabolic Method (PPM) for Gasdynamical Simulations", J. Comput. Phys., **54**, 174-201 (1984).