Fast Fourier Transform

Monday, 13 March 2023 10:14

$$H(f) = \int_{-\infty}^{\infty} h(t)\, e^{2\pi i f t}\, dt$$

$$h(t) = \int_{-\infty}^{\infty} H(f)\, e^{-2\pi i f t}\, df$$

Fourier Transform Equations

t in secs $\Longrightarrow$ f in cycles/sec Hz

x in meters $\Longrightarrow$ k in cycles/meter
wavenumber

$\omega \equiv 2\pi f$ – Physics → angular frequency

$$H(\omega) = \int h(t)\, e^{i\omega t}\, dt$$

$$h(t) = \frac{1}{2\pi}\int H(\omega)\, e^{-i\omega t}\, d\omega$$

QM & EM

Properties of FTs:

linear $\Longrightarrow$ $h(t) + g(t) \Longleftrightarrow H(f) + G(F)$

Symmetries

complex conj.

$h(t)$ real	$H(-f) = [H(f)]^*$
$h(t)$ imaginary	$H(-f) = -[H(f)]^*$
$h(t)$ odd $h(t) = -h(-t)$	$H(-f) = -H(f)$ or H is odd
$h(t)$ even $h(t) = h(-t)$	H is even

Speed up computation and reduce the storage

$$h(at) \Longleftrightarrow \frac{1}{|a|} H\left(\frac{f}{a}\right) \quad \text{"time scaling"}$$

$b \to \frac{1}{a}$ "frequency scaling"

$$\frac{1}{|b|} h\left(\frac{t}{b}\right) \Longleftrightarrow H(bf)$$

$$h(t - t_0) \Longleftrightarrow H(f)\, e^{2\pi i f t_0} \quad \text{"time shift"}$$

Convolution:

$$g * h \equiv \int_{-\infty}^{\infty} g(t')\, h(t - t')\, dt'$$

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$$\Longleftrightarrow \quad G(f)\,H(f)$$

Correlations

$$\mathrm{Corr}(g,h) \equiv \int_{-\infty}^{\infty} g(t'+t)\,h(t')\,dt'$$

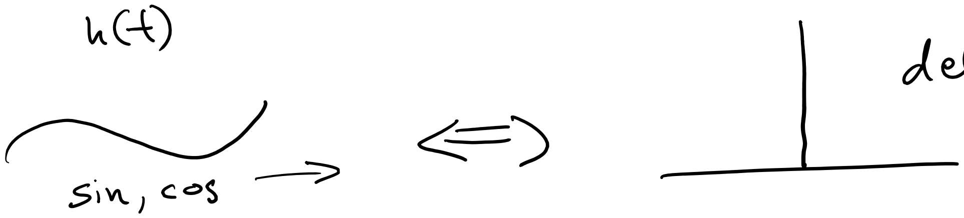
$$\Longleftrightarrow \quad G(f)\,H^*(f)$$

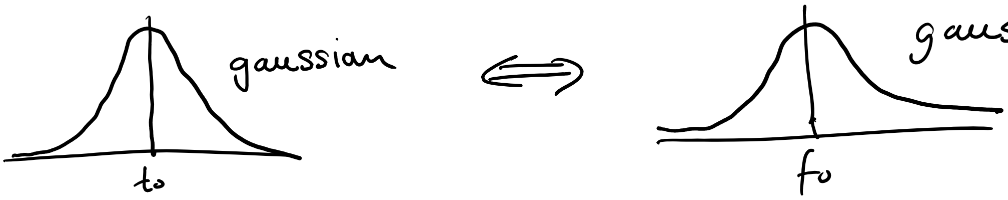
$$\mathrm{Corr}(g,g) \Longleftrightarrow |G(f)|^2$$

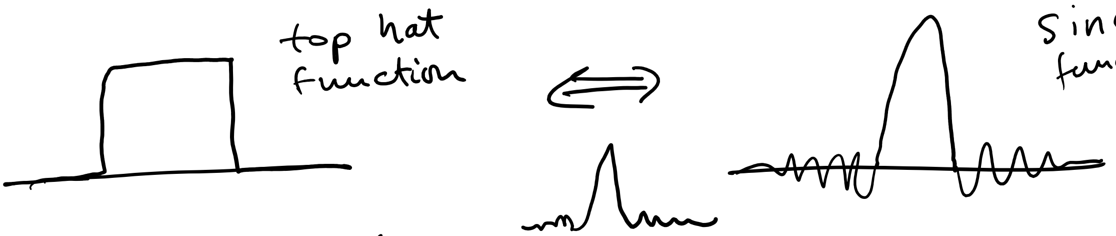
Parseval's Theorem

$$\text{Total Power} \equiv \int_{-\infty}^{\infty} |h(t)|^2\,dt = \int_{-\infty}^{\infty} |H(f)|^2\,df$$

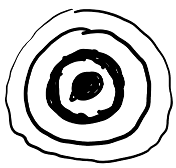
transformation is unitary.







$P_h(f)$



Airy Disc

Diffraction Limited image

Discrete Fourier Transform:

$$h_n = h(n\Delta) \qquad n = \ldots\ldots, -3, -2, -1, 0, 1, 2, \ldots$$

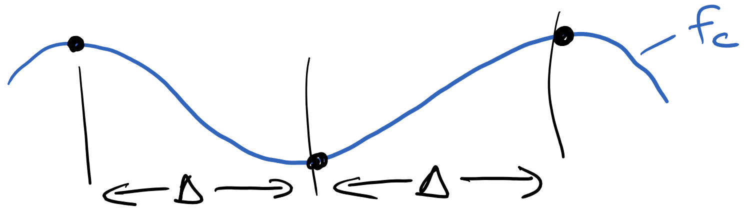
$\frac{1}{\Delta}$ sampling rate

special limiting

$\frac{1}{\Delta}$ sampling rate

For any Δ there is a special limiting frequency f_c called the Nyquist frequency

$$f_c = \frac{1}{2\Delta}$$



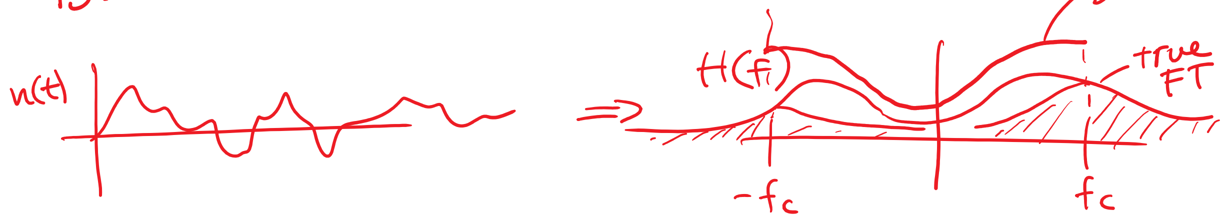
Critical sampling is 2 points per cycle.

$\boxed{\text{Sampling Theorem}}$

If continuous $h(t)$, sampled at Δ, happens to be <u>band-width limited</u> to frequencies smaller than f_c, i.e., $H(f) = 0$ for all $|f| > f_c$, then $h(t)$ is <u>completely determined</u> by its samples h_n!

$$h(t) = \Delta \sum_{n=-\infty}^{\infty} h_n \frac{\sin[2\pi f_c (t - n\Delta)]}{\pi (t - n\Delta)}$$

Not BW-limited:



ALIASING

can't remove it after the fact

$$\underline{y} = W_n \underline{x} \quad \text{linear transformation}$$

$$h_q \equiv x_q = h(t_q), \quad t_q = q\Delta$$

$\Rightarrow$ N input numbers $\rightarrow$ N independent output numbers

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We will assume N is even!

$H(f)$ @ discrete steps in

$$-f_c \leq f \leq f_c$$

$$f_n = \frac{n}{N\Delta}, \quad n = -\frac{N}{2}, \ldots, \frac{N}{2}$$

Looks like $N+1$ outputs!?

See below...

$\underline{y} = W_n \underline{x}$ looks like $\Theta(N^2)$

$$\omega_{p,q} = e^{(2\pi i/n)pq} = \omega^{pq} \qquad \omega \equiv e^{\left(\frac{2\pi i}{N}\right)}$$

☆ But there is lots of symmetry in W_n! We can exploit this to get to an $\Theta(N \log N)$ algorithm!

$$\omega_{p,n-q} = e^{(2\pi i \frac{n}{n})p} e^{-(2\pi i/n)pq}$$

$$e^{2\pi i \cdot p} \; e^{-(2\pi i/n)pq}$$

$$(1)^p$$

$$= 1 \cdot \omega^*_{p,q}$$

$$\omega_{p,q+\frac{n}{2}} = e^{(2\pi i/n)p(q+\frac{n}{2})}$$

$$= e^{\cancel{2}\pi i/\cancel{n} \cdot \frac{\cancel{n}}{\cancel{2}} \cdot p} e^{(2\pi i/n)pq}$$

$$= e^{i\pi p} \omega_{p,q}$$

$$= (-1)^p \omega_{p,q}$$

$$\omega_{p,q+\frac{n}{4}} = ??$$

$$y_p = \sum_{q=0}^{n-1} \omega^{pq} x_q \qquad \boxed{p = 0, \ldots, n-1}$$

$$= \sum_{q \text{ even}} \omega^{pq} x_q + (\text{odd})$$

$n/2 - 1$

$\underbrace{\qquad}_{q \text{ even}}$

$$= \sum_{r=0}^{n/2-1} \omega_n^{p(2r)} x_{2r} + \sum_{r=0}^{n/2-1} \omega_n^{p(2r+1)} x_{2r+1}$$

$$= \sum_{r=0}^{n/2-1} \omega_{\frac{n}{2}}^{pr} x_{2r} + \omega_n^{p} \sum_{r=0}^{n/2-1} \omega_{n/2}^{pr} x_{2r+1}$$

$$\underline{y} = W_{\frac{n}{2}} \underline{x}_{even} + \text{diag}(\underline{\omega}_n) W_{\frac{n}{2}} \underline{x}_{odd}$$

$$\text{diag}(\underline{\omega}_n) = \text{diag}(1, \omega, \omega^2, \omega^3, \dots, \omega^{n/2-1})$$

"twiddle factors"

$$y_{n/2} = W_{\frac{n}{2}} \underline{x}_{even} - \text{diag}(\underline{\omega}_n) W_{\frac{n}{2}} \underline{x}_{odd}$$

second half of y

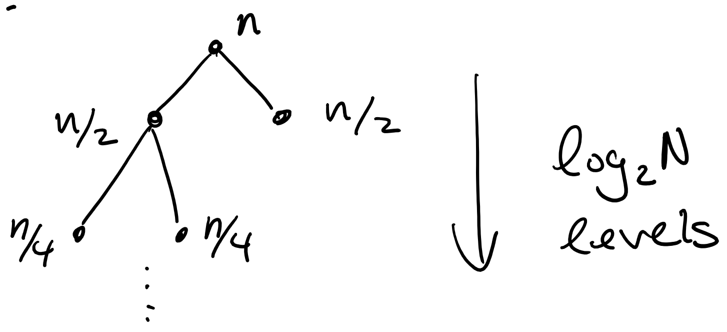
$y = W_n \underline{x}$ costs $8n^2$ float ops $(+, *)$

New cost $\underbrace{\boxed{8\left(\frac{n}{2}\right)^2}}_{\text{even}} + \underbrace{\boxed{6n}}_{\text{twiddle}} + \underbrace{\boxed{8\left(\frac{n}{2}\right)^2}}_{\text{odd}}$

$$= 4n^2 + 6n$$

Almost a factor of 2 cheaper

$$W_2 = ?$$



W_2 const ops

$O(N \log N)$ method.

Cooley-Tuckey Method.