Exam Doodle → sent to you!

60% Oral + 40% exercises!

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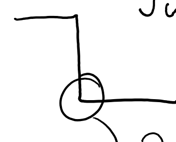
Finite Difference Methods to approximate derivatives

$$\frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} = 0$$

operators on the grid

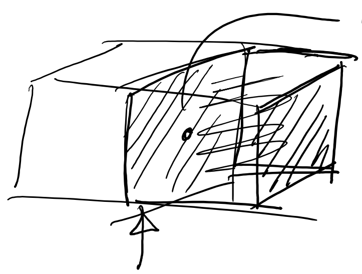
Conservation of Mass



Jump Shock-wave propagating

$\frac{\partial \rho}{\partial x}$ is not good here!

Integral equations makes sense here.

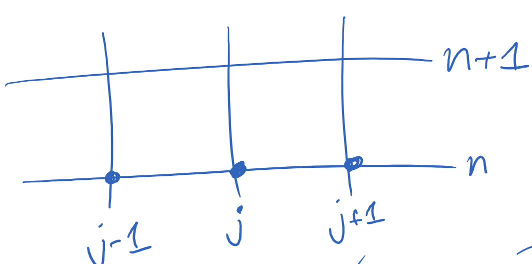



how much mass flows through the surfaces

$$\rho_j^{n+1} = \rho_j^n + \frac{\Delta t}{\Delta x}\left[f_{j-\frac{1}{2}} - f_{j+\frac{1}{2}}\right]$$

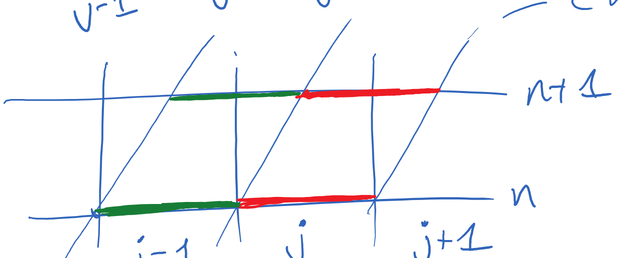
Integration over all fluxes should be exact. ‖ Approximate the fluxes.

For linear advection $f(\rho) = a\cdot\rho \quad a \geq 0$




Finite Difference Picture

Characteristics with slope a




Finite Volume

$$\rho_j^{n+1} = \underbrace{\boxed{\left(\frac{a\cdot\Delta t}{\Delta x}\right)}}_{c}\rho_{j-1}^n + \underbrace{\boxed{\left(1-\frac{a\cdot\Delta t}{\Delta x}\right)}}_{(1-c)}\rho_j^n$$

Finite Volume

Godunov Method

$$\rho_j^{n+1} = c\rho_{j-1}^n + (1-c)\rho_j^n \qquad \boxed{a \geq 0}$$

$$\boxed{\rho_j^{n+1} - \rho_j^n + c\left(\rho_j^n - \rho_{j-1}^n\right) = 0}$$

"CIR" scheme

This is exactly the same as the 1st order upwind method. In the case of linear advection.

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$$\frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} = 0 \implies$$ turns out that a numerical diffusion term: $\frac{\partial^2 \rho}{\partial x^2}$ !

$$\frac{\rho_j^{n+1} - \rho_j^n}{\Delta t} + a\,\frac{\rho_{j+1}^n - \rho_{j-1}^n}{2\Delta x} = 0$$

Recall: this is unstable.

Taylor expand $\rho$ in time to 2nd order:

$$\rho_j^{n+1} = \cancel{\rho_j^n} + \Delta t\left(\frac{\partial \rho}{\partial t}\right) + \frac{\Delta t^2}{2}\left(\frac{\partial^2 \rho}{\partial t^2}\right) + \cdots$$

Taylor expand $\rho$ is space to 2nd order:

$$\rho_{j+1}^n = \cancel{\rho_j} + \Delta x\left(\frac{\partial \rho}{\partial x}\right) + \frac{\Delta x^2}{2}\left(\frac{\partial^2 \rho}{\partial x^2}\right) + \cdots$$

$$\rho_{j-1}^n = \cancel{\rho_j} - \Delta x\left(\frac{\partial \rho}{\partial x}\right) + \frac{\Delta x^2}{2}\left(\frac{\partial^2 \rho}{\partial x^2}\right) + \cdots$$

Plug them in!

$$\frac{\cancel{\Delta t}\left(\frac{\partial \rho}{\partial t}\right) + \frac{\Delta t^2}{2}\left(\frac{\partial^2 \rho}{\partial t^2}\right)}{\;} + a\,\frac{\left(\cancel{2\Delta x}\left(\frac{\partial \rho}{\partial x}\right)\right)}{\cancel{2\Delta x}} = 0$$

$$\frac{\cancel{\Delta t}\left(\frac{\partial \rho}{\partial t}\right) + \frac{\Delta t^2}{2}\left(\frac{\partial^2\rho}{\partial t^2}\right)}{\cancel{\Delta t}} + a\,\underbrace{\frac{\left(\cancel{2\Delta x}\left(\frac{\partial\rho}{\partial x}\right)\right)}{\cancel{2\Delta x}}} = 0$$

$$\frac{\partial\rho}{\partial t} + a\frac{\partial\rho}{\partial x} = -\frac{\Delta t}{2}\left(\underset{?}{\underline{\underline{\frac{\partial^2\rho}{\partial t^2}}}}\right) + O(\Delta t^2, \Delta x^2)$$

$$\frac{\partial\rho}{\partial t} + a\frac{\partial\rho}{\partial x} = 0$$

$$\frac{\partial^2\rho}{\partial t^2} + a\frac{\partial}{\partial x}\underbrace{\left(\frac{\partial\rho}{\partial t}\right)}_{-a\frac{\partial\rho}{\partial x}} = 0$$

$$\frac{\partial^2\rho}{\partial t^2} = -a^2\frac{\partial^2\rho}{\partial x^2}$$

Diffusion term

$$\frac{\partial\rho}{\partial t} + a\frac{\partial\rho}{\partial x} = \boxed{-a^2\frac{\Delta t}{2}}\left(\frac{\partial^2\rho}{\partial x^2}\right)$$

D is negative is not "diffusion" but amplification!

Unstable

$\boxed{\text{Modified Equation}}$ $$\frac{\partial\rho}{\partial t} + a\frac{\partial\rho}{\partial x} = -a^2\frac{\Delta t}{2}\frac{\partial^2\rho}{\partial x^2}$$ Advection-Diffusion Equation

What is the modified equation for the C.I.R. Scheme.

## 2-D Advection

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0$$

$$\underline{u} = \langle a, b \rangle \qquad a, b > 0$$

$$\frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} + b\frac{\partial \rho}{\partial y} = 0$$

$1^{st}$ order upwind to solve (C.I.R.):

$$\frac{\rho_{jl}^{n+1} - \rho_{jl}^{n}}{\Delta t} + a\,\frac{\rho_{jl}^{n} - \rho_{j-1\,l}^{n}}{\Delta x} + b\,\frac{\rho_{jl}^{n} - \rho_{j\,l-1}^{n}}{\Delta y} = 0$$

Stability analysis shows that

$C_a > 0$ $\qquad C_a = \frac{a\Delta t}{\Delta x}$ $\qquad \boxed{C_a + C_b \le 1}$

$C_b > 0$

Sum of the Courant numbers in x and y must be less than 1!

Stricter criteria

Modified Equation for this is interesting:

$$\frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} + b\frac{\partial \rho}{\partial y} = \frac{a\Delta x}{2}(1 - C_a)\frac{\partial^2 \rho}{\partial x^2}$$

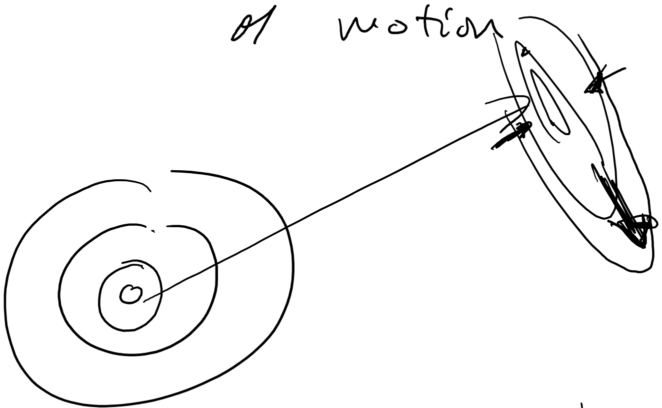
$$+ \frac{b\Delta y}{2}(1 - C_b)\frac{\partial^2 \rho}{\partial y^2}$$

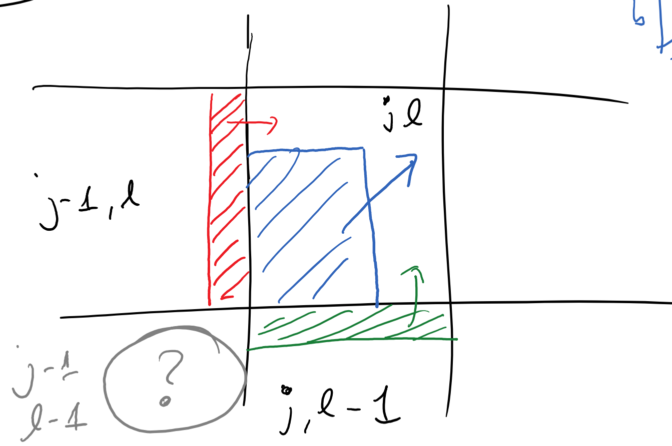
$$\boxed{- ab\Delta t \frac{\partial^2 \rho}{\partial x \partial y}}$$

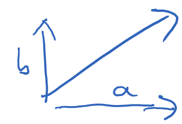
a new term gets added!

Won't preserve shape!

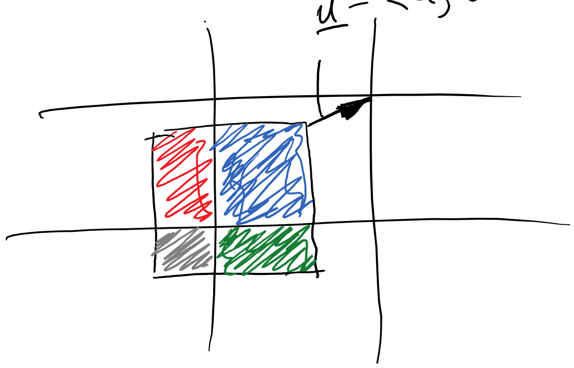
There is an extra diffusion in the tangential direction to the motion, and anti-diffusion in the direction of motion






## Corner Transport Upwind Method




$$
\begin{aligned}
\rho_{jl}^{n+1} = {} & \boxed{(1-C_a)(1-C_b)}\,\rho_{jl}^{n} \\
& + \boxed{C_a(1-C_b)}\,\rho_{j-1\,l}^{n} \\
& + \boxed{C_b(1-C_a)}\,\rho_{j\,l-1}^{n} \\
& + \boxed{C_a C_b}\,\rho_{j-1\,l-1}^{n}
\end{aligned}
$$

Stability is (slightly) better:

$$0 \le C_a < 1, \quad 0 \le C_b < 1$$

Nice 2-step method to do this:

Nice 2-step method ...

$$\boxed{\begin{aligned} \rho_{jl}^{*} &= (1 - C_a)\,\rho_{jl}^{n} + C_a\,\rho_{j-1\,l}^{n} \\ \rho_{jl}^{n+1} &= (1 - C_b)\,\rho_{jl}^{*} + C_b\,\rho_{j\,l-1}^{*} \end{aligned}}$$

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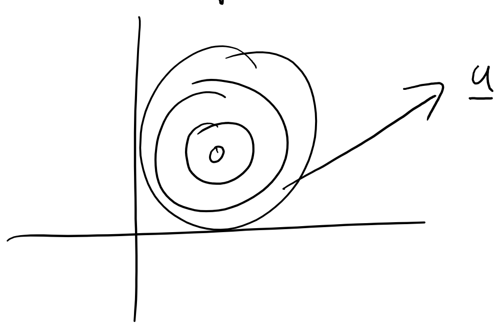
2-D transport




$$\rho(x,y) = A \exp\left[ -\left( \frac{(x-x_0)^2}{2\sigma_x^2} + \frac{(y-y_0)^2}{2\sigma_y^2} \right) \right]$$

$$\sigma_x = \sigma_y$$

Initial Condition

2-D CIR vs CTU methods