Symplectic Integrators

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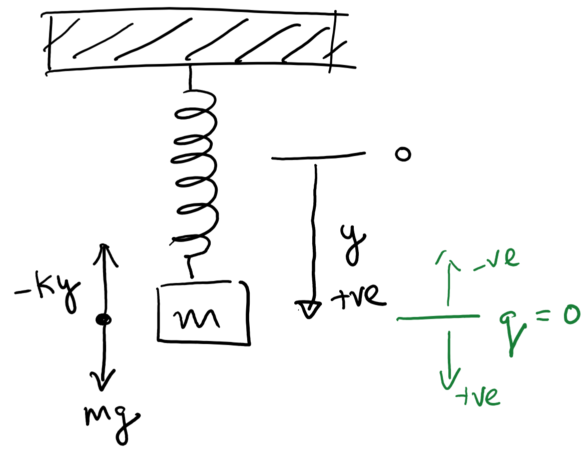
ODE $\rightarrow$ Runge-Kutta "Blackbox"

Symplectic Integrators $\Rightarrow$ try to preserve something or use information about the solution. e.g., Periodic
Consv. of Energy

Harmonic Oscillator:



$$F = mg - ky$$

to make it easier $m = 1 \quad k = 1$

$$F = g - y$$

Define $q=0$ where $F=0$

$$q = y - g = -F$$

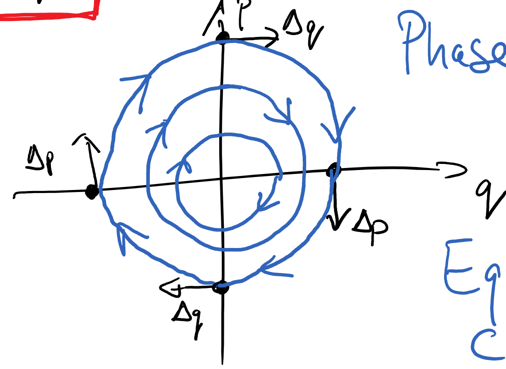
Newton's Law $F = ma = m\ddot{y}$ $\qquad \dot{y} = \frac{dy}{dt} \leftarrow$ velocity

$$\boxed{\ddot{q} = \ddot{y} = F = -q} \longrightarrow \ddot{q} = -q$$

Introduce Momentum $p = mv = m\dot{y} = \dot{q}$

$$\boxed{\begin{aligned} \dot{q} &= p \\ \dot{p} &= -q \end{aligned}}$$

Harmonic Oscillator



Phase Diagram

Equation for the circle

$E = \frac{1}{2}mv^2 + ky^2$

circle

$r^2 = p^2 + q^2$

Should be conserved for all time!

Related to the Energy of the system.

$$H = \frac{1}{2}(p^2 + q^2)$$

Hamiltonian

ODE is easy

$$\dot{q} = \frac{\partial H}{\partial p}$$

$$\dot{p} = -\frac{\partial H}{\partial q}$$

Seperable Hamiltonian

$$H = T(p) + U(q)$$

if this is the case, then we can conserve lots of things numerically <u>exactly</u>.

What happens with Forward-Euler method

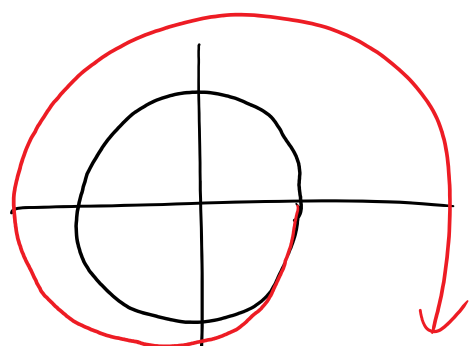
$$q_{n+1} = q_n + h p_n$$

$$p_{n+1} = p_n - h q_n$$

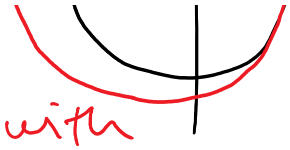
$$q_{n+1}^2 + p_{n+1}^2 = (1 + h^2)(q_n^2 + p_n^2)$$

$$r_{n+1}^2 = (1 + h^2) r_n^2$$

Exponentially diverging away



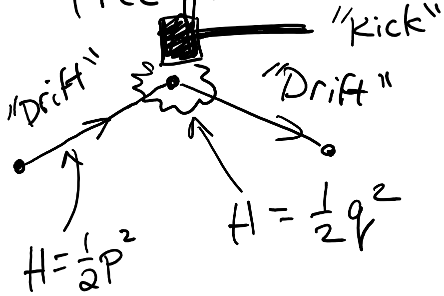
diverging away



This will happen with <u>any</u> black box method.

Idea is to evolve the system with each of the seperable parts of the Hamiltonian on their own and in a certain pattern.

$T(p) \rightarrow$ "free particle" $U(q) \rightarrow$ only changes p. instantaneous impulse



Leap-Frog, Strömer-Verlet Method

For Harmonic Oscillator:

$$\boxed{q_{n+\frac{1}{2}}} = q_n + \frac{1}{2} h p_n \quad \text{"half drift"}$$

$$\boxed{p_{n+1}} = p_n - h \boxed{q_{n+\frac{1}{2}}} \quad \text{"Kick"}$$

$$q_{n+1} = q_{n+\frac{1}{2}} + \boxed{\frac{1}{2} h p_{n+1}} \quad \text{"half drift"}$$

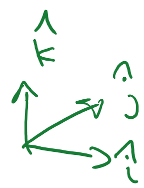
$$H = \underset{\text{"}\underline{v}\text{"}}{T(p)} + \underset{\text{"}\underline{x}\text{"}}{U(q)}$$

$q, \underline{x}$ \ position and
$p, \underline{v}$ / velocity vectors

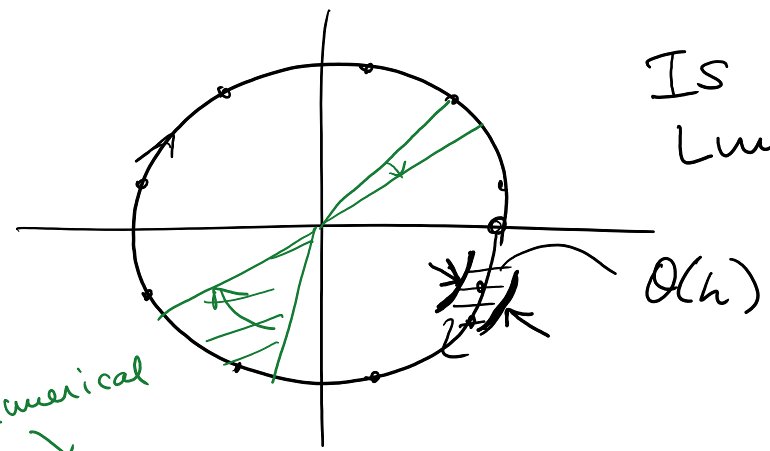
$$\underline{x}_{\frac{1}{2}} = \underline{x}_0 + \frac{1}{2} h \underline{v}_0$$

$$\underline{v}_1 = \underline{v}_0 + h\left(-\nabla U(\underline{x}_{\frac{1}{2}})\right)$$

$$\underline{x}_1 = \underline{x}_{\frac{1}{2}} + \frac{1}{2} h \underline{v}_1$$



$$\nabla U = \frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j} + \frac{\partial U}{\partial z}\hat{k}$$



Is this a "free Lunch"?

$O(h^3)$ truncation error

Where is the error.

numerical

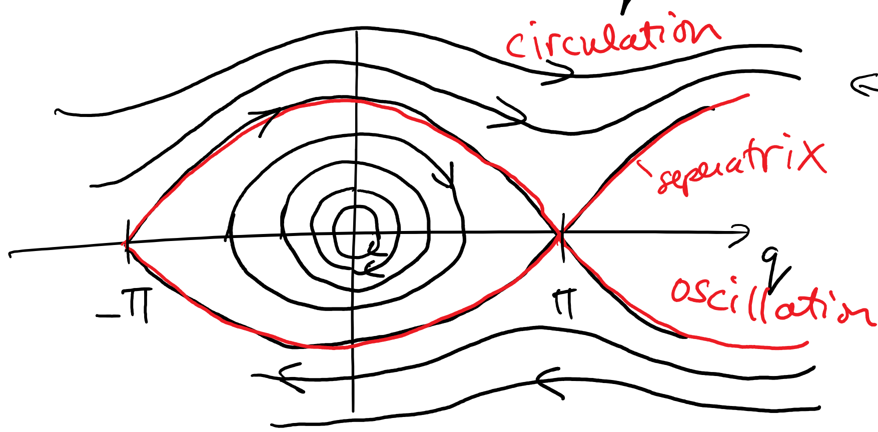
$\widetilde{\text{Period}} \neq \text{Period}$

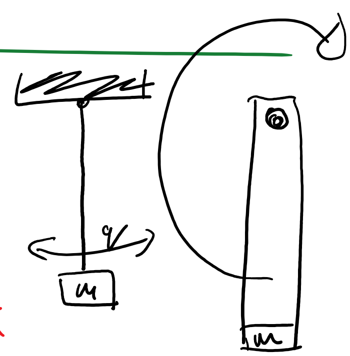
Conserves ✓

Periodic ✓

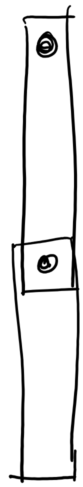
$$H = \frac{1}{2}p^2 - \varepsilon \cos q$$

an angle





Simple pendulum



Double Pendulum

$\Rightarrow$ Chaotic