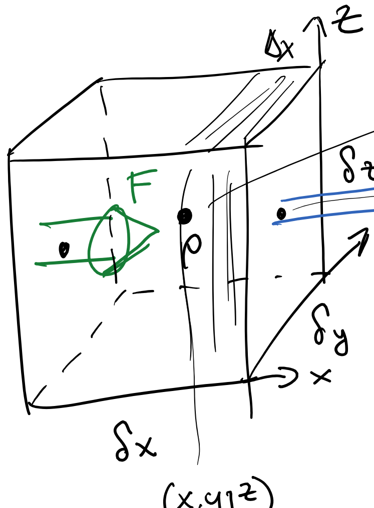
Very small cube



density in the center of the very small cube $\delta \to$ very small

face center +ve x-direction

$$\rho(x+\tfrac{1}{2}\delta x, t) \approx \rho(x,t) + \tfrac{1}{2}\delta x \frac{\partial \rho}{\partial x}\Big|_{x,t}$$

Flux $\equiv$ material flowing through the surface per unit time

$$\rho \frac{\Delta x \, \delta y \delta z}{\Delta t} = \underbrace{\rho u}\, \underbrace{\delta y \delta z}$$

velocity in the x-direction

Want: ρu at the face:

ρu at $x+\frac{1}{2}\delta x$ is $\approx \rho u + \frac{1}{2}\delta x \frac{\partial(\rho u)}{\partial x}\Big|_{x,t}$

Flux through the surface is then given by,

$$\left\{\rho u + \frac{1}{2}\delta x \frac{\partial(\rho u)}{\partial x}\Big|_{x,t}\right\}\delta y \delta z$$

Flux through the opposite face is then,

$$\left\{\rho u - \frac{1}{2}\delta x \frac{\partial(\rho u)}{\partial x}\Big|_{x,t}\right\}\delta y \delta z$$

Now : the flux <u>into</u> the cube the cube is given by the difference of these 2 Fluxes;

$$-1\,\delta x \frac{\partial(\rho u)}{\partial x}\Big|_{x,t}\delta y \delta z = -\frac{\partial(\rho u)}{\partial x}\delta V$$

The change in mass in the cube is given by,

$$\frac{\partial \rho}{\partial t} \cdot \delta V .$$

If mass is conserved this has to be a result of material flowing into the cube (flux):

$$\frac{\partial \rho}{\partial t} \cancel{\delta V} = - \frac{\partial (\rho u_x)}{\partial x} \cancel{\delta V}$$

only in the x-direction!

$$- \frac{\partial (\rho u_y)}{\partial y} + - \frac{\partial (\rho u_z)}{\partial z}$$

$$\equiv - \nabla \cdot (\rho \underline{u})$$

$$\frac{\partial \rho}{\partial t} = - \nabla \cdot (\rho \underline{u})$$

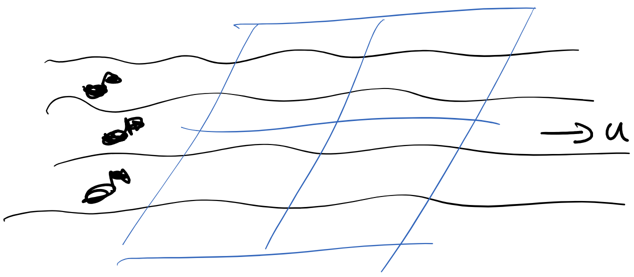
$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0}$$ Hyperbolic PDE

Conservation of Mass

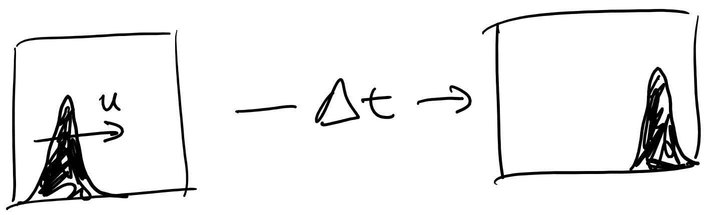
+ Conservation of Energy

+ Conservation of Momentum

Linear Advection Equation



River

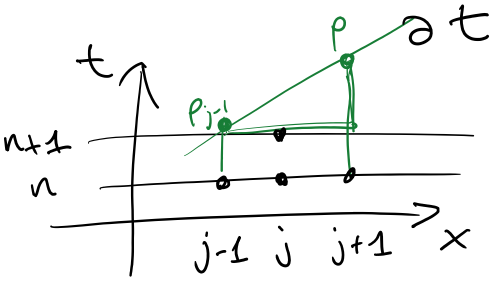


$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} = 0$$

How do you treat this numerically?

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} = 0$$

How do you ... this numerically?



$$\left.\frac{\partial \rho}{\partial t}\right|_j \approx \frac{\rho_j^{(n+1)} - \rho_j^{(n)}}{\Delta t}$$

$$\left.\frac{\partial \rho}{\partial x}\right|_j \approx \frac{\rho_{j+1}^{(n)} - \rho_{j-1}^{(n)}}{2\Delta x}$$

$$\boxed{\rho_j^{(n+1)} = \rho_j^{(n)} - \frac{1}{2}C\left(\rho_{j+1}^{(n)} - \rho_{j-1}^{(n)}\right) \qquad C \equiv \frac{\Delta t\, u}{\Delta x}}$$

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von Neumann Stability Analysis

$$\rho_j^{(n)} = A^n e^{ij\theta} \qquad i = \sqrt{-1}$$

$$A^{n+1} e^{ij\theta} = A^n e^{ij\theta} - \frac{1}{2} C A^n \left(e^{i(j+1)\theta} - e^{i(j-1)\theta}\right)$$

cancel $A^n e^{ij\theta}$ everywhere

$$A = 1 - C\left(\frac{e^{i\theta} - e^{-i\theta}}{2}\right)$$

$$\boxed{A = 1 - iC\sin\theta}$$

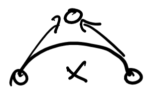
$$A^*A = |A|^2 = 1 + C^2\sin^2\theta > 1$$

Disaster: the method is useless!

LAX METHOD: Replace $\rho_j^{(n)}$ with $\frac{1}{2}\left(\rho_{j+1}^{(n)} + \rho_{j-1}^{(n)}\right)$

$$\rho_j^{(n+1)} = \frac{1}{2}\left(\rho_{j+1}^{(n)} + \rho_{j-1}^{(n)}\right) - \frac{1}{2}C\left(\rho_{j+1}^{(n)} - \rho_{j-1}^{(n)}\right)$$

$$A = \cos\theta - iC\sin\theta$$



Stable when $|c| \lesssim 1$

Courant Condition : CFL condition

$$|u| < \frac{\Delta x}{\Delta t}$$

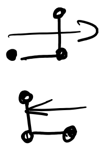
Physical Velocity

Grid Velocity

Physical information must not travel faster than the maximum velocity "of the" grid.

This is the best you can do.

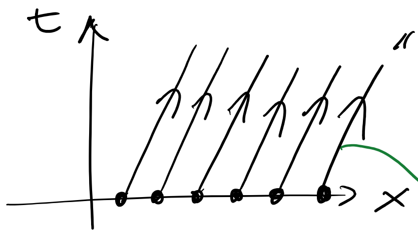
Upwind Schemes :



$$\rho_j^{(n+1)} = \rho_j^{(n)} - c\left(\rho_j^{(n)} - \rho_{j-1}^{(n)}\right) \quad \text{for } u > 0$$

$$\qquad\qquad - c\left(\rho_{j+1}^{(n)} - \rho_j^{(n)}\right) \quad \text{for } u < 0$$

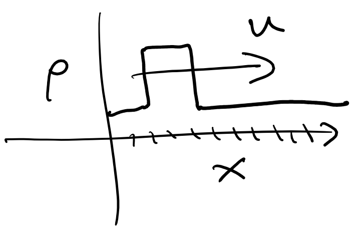
First order upwind scheme

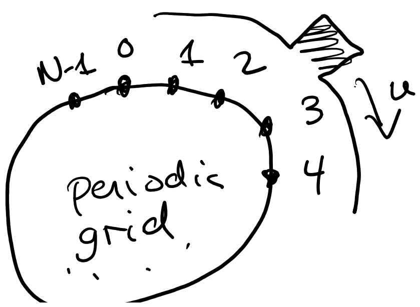


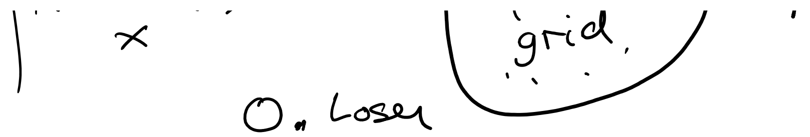
"Ducks" move along parallel lines.

Characteristics of the solution, or equation (ODEs describe these)

1-D Linear advection







0. Loser
1. LAX
2. CIR 1^{ST} order upwind
3. LAX-WENDROFF METHOD (upwind)

$$\rho_j^{(n+1)} = \frac{1}{2} c(1+c)\rho_{j-1}^{(n)} + (1-c^2)\rho_j^{(n)} - \frac{1}{2} c(1-c)\rho_{j+1}^{(n)}$$

for $u>0$