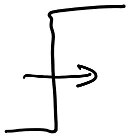
Finite Volume Methods

Monday, 5 December 2022 13:02

Finite Difference Method

$$\frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} = 0$$

operators on a grid

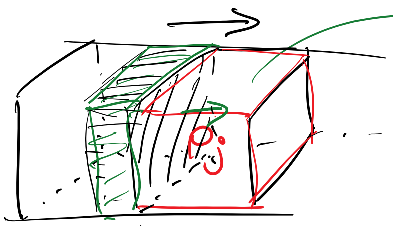


"Jump"

$\frac{\partial \rho}{\partial x}$ is not well defined

Conservation Laws lead to Integral Equations

how much mass flows through the surfaces



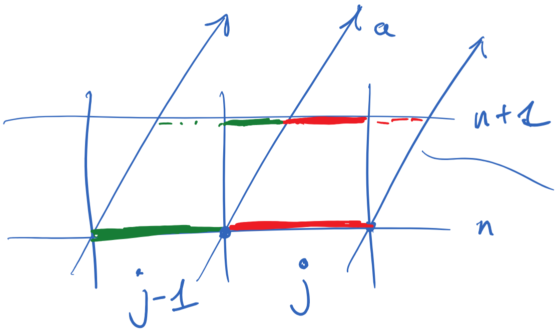
flux

$$\rho_j^{n+1} = \rho_j^n + \frac{\Delta t}{\Delta x}\left[\underbrace{f_{j-\frac{1}{2}} - f_{j+\frac{1}{2}}}_{\text{Approximate These Fluxes}}\right]$$

Integration over all fluxes should be exactly 0.

Linear Advection $\quad f(\rho) = a\cdot\rho \quad a \geq 0$

Finite Volume Method



Characteristics with slope a

$$\rho_j^{n+1} = \underbrace{\boxed{\left(\frac{a\cdot\Delta t}{\Delta x}\right)}}_{c}\rho_{j-1}^n + \underbrace{\boxed{\left(1-\frac{a\Delta t}{\Delta x}\right)}}_{(1-c)}\rho_j^n$$

Godunov's Method

$$\rho_j^{n+1} = c\,\rho_{j-1}^n + (1-c)\rho_j^n \quad a \geq 0$$

$$\boxed{\rho_j^{n+1} - \rho_j^n + c\left(\rho_j^n - \rho_{j-1}^n\right) = 0}$$

$$\rho_j^{n+1} - \rho_j^n + C(\rho_j^n - \rho_{j-1}^n) = 0$$

1^{st} order upwind method

Same in the case of linear advection.

Modified Equation

useless Method

$$\frac{\rho_j^{n+1} - \rho_j^n}{\Delta t} + a\,\frac{\rho_{j+1}^n - \rho_{j-1}^n}{2\Delta x} = 0$$

Recall: this is unstable

Taylor expand ρ_j^{n+1} to 2^{nd} order:

(in time) $$\rho_j^{n+1} = \cancel{\rho_j^n} + \Delta t\left(\frac{\partial \rho}{\partial t}\right) + \frac{\Delta t^2}{2}\left(\frac{\partial^2 \rho}{\partial t^2}\right) + \dots$$

Taylor expand $\rho_{j+1}^n, \rho_{j-1}^n$ to 2^{nd} order in x

$$\rho_{j+1}^n = \cancel{\rho_j^n} + \Delta x\left(\frac{\partial \rho}{\partial x}\right) + \cancel{\frac{\Delta x^2}{2}\left(\frac{\partial^2 \rho}{\partial x^2}\right)} + \dots$$

$$\rho_{j-1}^n = \cancel{\rho_j^n} - \Delta x\left(\frac{\partial \rho}{\partial x}\right) + \cancel{\frac{\Delta x^2}{2}\left(\frac{\partial^2 \rho}{\partial x^2}\right)} + \dots$$

$$\frac{\cancel{\Delta t}\left(\frac{\partial \rho}{\partial t}\right) + \frac{\Delta t^2}{2}\left(\frac{\partial^2 \rho}{\partial t^2}\right)}{\cancel{\Delta t}} + a\,\frac{\left(\cancel{2\Delta x}\left(\frac{\partial \rho}{\partial x}\right)\right)}{\cancel{2\Delta x}} = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} = -\frac{\Delta t}{2}\left(\frac{\partial^2 \rho}{\partial t^2}\right)$$

?

$$\boxed{\frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} = 0}$$

$$\frac{\partial^2 \rho}{\partial t^2} + a\frac{\partial}{\partial x}\left(\underbrace{\frac{\partial \rho}{\partial t}}_{-a\frac{\partial \rho}{\partial x}}\right) = 0$$

$$\frac{\partial^2 \rho}{\partial t^2} = -a^2\frac{\Delta t}{2}\left(\frac{\partial^2 \rho}{\partial x^2}\right)$$

$$\frac{\partial \rho}{\partial t} + \boxed{a\frac{\partial \rho}{\partial x}} = \boxed{-a^2\Delta t}\left(\frac{\partial^2 \rho}{\partial x^2}\right)$$

$$\frac{\partial \rho}{\partial t} + \underbrace{\boxed{a\frac{\partial \rho}{\partial x}}}_{=0} = \underbrace{\boxed{-a^2\frac{\Delta t}{2}}}_{=D}\left(\frac{\partial^2 \rho}{\partial x^2}\right)$$

it is negative !

unstable

Diffusion

Ex. What is the modified equation for C.I.R. Scheme (1st order upwind)

2-D Advection

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0$$

$\underline{u} = \langle a, b \rangle \qquad a, b > 0$

$$\frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} + b\frac{\partial \rho}{\partial y} = 0$$

1st order upwind to solve this

$$\frac{\rho_{je}^{n+1} - \rho_{je}^{n}}{\Delta t} + a\,\frac{\rho_{je}^{n} - \rho_{j-1e}^{n}}{\Delta x} + b\,\frac{\rho_{je}^{n} - \rho_{je-1}^{n}}{\Delta y} = 0$$

Stability Analysis shows that

$C_a > 0 \qquad C_a = \dfrac{a\Delta t}{\Delta x} \qquad \boxed{C_a + C_b \leq 1}$

$C_b > 0$

Sum of the 2 Courant numbers in x and y must be less than 1.

Modified Equation for this is interesting:

$$\frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} + b\frac{\partial \rho}{\partial y} = \frac{a\Delta x}{2}(1 - C_a)\frac{\partial^2 \rho}{\partial x^2}$$

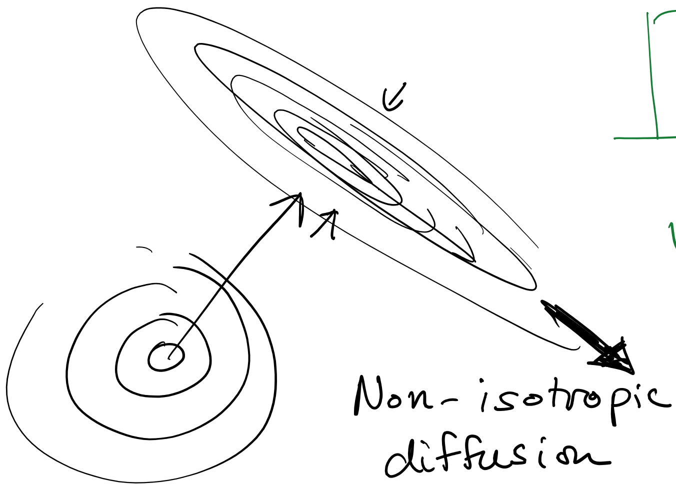
$+ \frac{b\Delta x}{}(1 - C\ldots)\frac{\partial^2 \rho}{}$

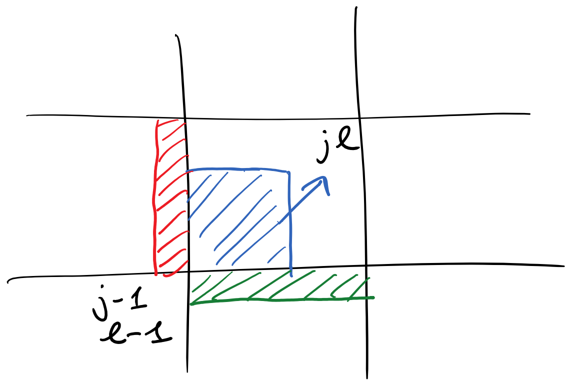
$\frac{\ }{2}(1-c_b)\frac{\ }{\partial y^2}$

$$\boxed{-ab\Delta t\frac{\partial^2\rho}{\partial x\partial y}}$$

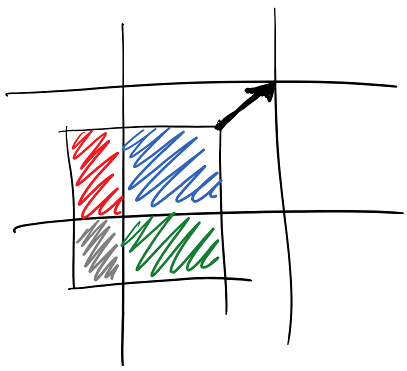
Cross derivative term.

Won't preserve shape





Corner Transport upwind Method



$$\rho_{jl}^{n+1} = (1-c_a)(1-c_b)\rho_{jl}^{n} + c_a(1-c_b)\rho_{j-1,l}^{n} + c_b(1-c_a)\rho_{jl-1}^{n} + c_a c_b \rho_{j-1\,l-1}^{n}$$

Nice 2-step Method

$$\boxed{\begin{aligned}\rho_{jl}^{*} &= (1-c_a)\rho_{jl}^{n} + c_a\rho_{j-1\,l}^{n}\\ \rho_{jl}^{n+1} &= (1-c_b)\rho_{jl}^{*} + c_b\rho_{jl-1}^{*}\end{aligned}}$$

Stability is better as well

$$0 \le c_a < 1\ ,\quad 0 \le c_b < 1$$

$$\rho(x,y) = A \exp\left[-\left(\frac{(x-x_0)^2}{2\sigma_x^2} + \frac{(y-y_0)^2}{2\sigma_y^2}\right)\right]$$

$$\sigma_x = \sigma_y$$

2-D C.I.R. vs CTU methods