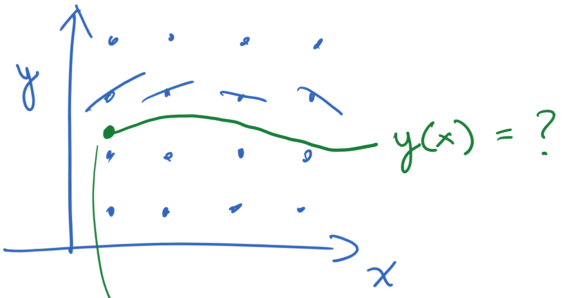
Ordinary Differential Equations

Monday, 17 October 2022 12:58

$$\frac{dy(x)}{dx} = \underline{f}(x, y)$$



x_{start}
y_{start}

Boundary condition

$$y_i(x_{start}) = y_i^{start}$$ Dirchlet B.C.

or

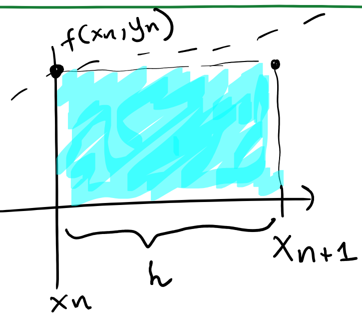
$$\frac{dy_i}{dx}(x_{start}) = m_i^{start}$$ Slope is given somewhere

Von Neumann type B.C.

Analytical is often hard
Numerical is easy.

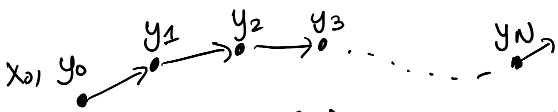
N-equations → Need N BCs!

$$\int_{y_n}^{y_{n+1}} dy = \int_{x_n}^{x_{n+1}} f(x, y)\, dx$$



$$\boxed{y_{n+1} - y_n = h \cdot f(x_n, y_n)}$$

Forward Euler Method



$$y(x) \approx \underset{x_0}{y_0},\ \underset{x_0+h}{y_1},\ y_2,\ \cdots\ \underset{x_0+Nh}{y_N}$$

Error depends on the stepsize h.

Truncation Error: error associated with the algorithm or method, and not the precision of the floating point calculations (Round off Error)

$$0.00134\ldots 36\underbrace{7}_{\text{Round off}}$$

Truncation Error   Round off error

Error of Forward Euler:

$$\int_{x_n}^{x_{n+1}} f(x,y)\,dx$$

Taylor expand

$$f(x,y(x)) = f(x, y(x_n+h))$$

$$= f\Big(x, y_n + \underbrace{h\cdot \frac{dy}{dx}\Big|_{x_n}}_{\text{small}}\Big)$$

Taylor expand again the function f

$$f(x,y(x)) \approx f(x,y_n) + h\cdot \boxed{\frac{dy}{dx}\Big|_{x_n}} \cdot f'(x,y_n)$$

slope at x_n $= f(x_n, y_n)$

$$\int_{x_n}^{x_{n+1}} f(x,y(x))\,dx \mathrel{\hat{=}} \int_{x_n}^{x_{n+1}} \left[f(x,y_n) + h\cdot f(x_n,y_n)\cdot f'(x,y_n)\right] dx$$

fix the function values at our initial condition at x_n

$$y_{n+1} - y_n \mathrel{\hat{=}} h\cdot f(x_n, y_n) + \underbrace{\boxed{h^2} f(x_n,y_n)\cdot f'(x_n,y_n)}_{\mathcal{O}(h^2)}$$

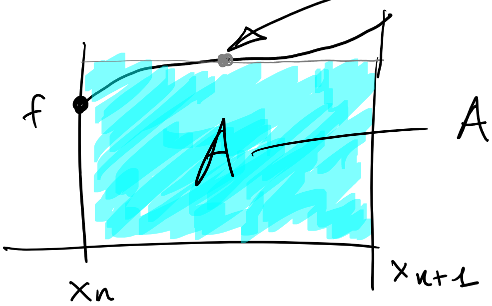
Local Error

$$\frac{X}{h} = N_{steps}$$

Global Error: Error over interval X is $N_{steps}\cdot \mathcal{O}(h^2) = \mathcal{O}(h)$

Very poor

Can we do better?



$f(x_{n+1/2}, y_{n+1/2})$

$$A = h\cdot f(x_{n+\frac{1}{2}}, \underline{y_{n+1/2}})$$

?

$$y_{n+\frac{1}{2}} = y_n + \frac{h}{2} f(x_n, y_n)$$

$$y_{n+\frac{1}{2}} = y_n + \frac{h}{2} f(x_n, y_n)$$

$$y_{n+1} - y_n = h \cdot f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right)$$

2 Evaluations of the right hand side functions $\underline{f}$!

Local Error : $O(h^3)$
Global Error : $O(h^2)$

Higher order method → price is more Evaluations
gain is less steps needed

Midpoint Runge-Kutta

4^{th} order R-K method

$$\underline{k_1} = h \cdot \underline{f}(x_n, y_n)$$

$$\underline{k_2} = h \cdot \underline{f}\left(x_n + \frac{h}{2}, y_n + \frac{\underline{k_1}}{2}\right)$$

$$\underline{k_3} = h \cdot \underline{f}\left(x_n + \frac{h}{2}, y_n + \frac{\underline{k_2}}{2}\right)$$

$$\underline{k_4} = h \cdot \underline{f}(x_n + h, y_n + \underline{k_3})$$

$$y_{n+1} = y_n + \frac{\underline{k_1}}{6} + \frac{\underline{k_2}}{3} + \frac{\underline{k_3}}{3} + \frac{\underline{k_4}}{6} + O(h^5)$$

Explicit Methods.

Predator - Prey Behaviour
Foxes and Mice
f m

Lotka-Voltera Model (1920)
without foxes we want the mice to grow exponentially

$$\underline{\Delta m} = k_{..} \cdot \Delta t$$

$$\frac{\Delta m}{m} = k_m \cdot \Delta t$$

↖ constant birth rate

but if the foxes are around the population reduces proportional to the number of foxes

$$\frac{\Delta m}{m} = k_m \cdot \Delta t - k_{mf} \cdot f \, \Delta t$$

$$\frac{dm}{dt} \approx \frac{\Delta m}{\Delta t} = \left(k_m \cdot m - k_{mf} \cdot \underbrace{m \cdot f}_{}\right)$$

$\propto$ number of encounters of foxes and mice

what about the foxes?

$$\frac{\Delta f}{f} = -k_f \, \Delta t$$

↖ exponential death rate for foxes.

$$\frac{\Delta f}{f} = -k_f \, \Delta t + k_{fm} \, m \, \Delta t$$

↖ birth rate proportional to the number of mice

$$\boxed{\begin{aligned} \frac{df}{dt} &= -k_f \cdot f + k_{fm} \, f \cdot m \\ \frac{dm}{dt} &= k_m \, m - k_{mf} \cdot m \cdot f \end{aligned}}$$

Solve this given some initial populations.

$k_m = 2$

$k_{mf} = 0.02$

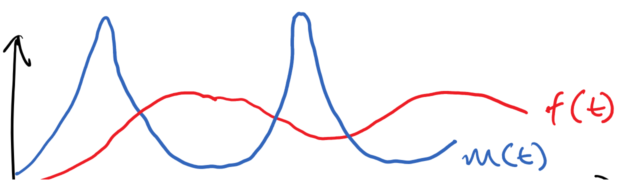
$k_{fm} = 0.01$

$k_f = 1.06$

IC $\quad m(0) = 100$

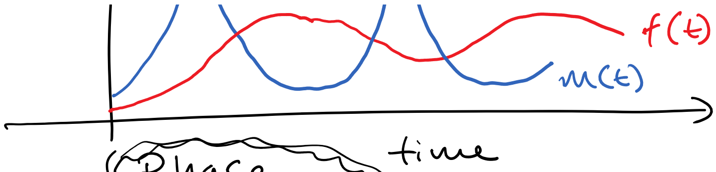
$\quad f(0) = 15$

$y = \langle m(t), f(t) \rangle$



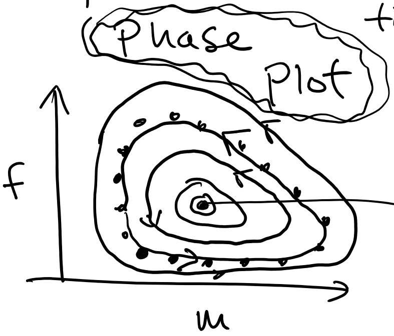
Use Forward-Euler method

Use Midpoint RK



use Midpoint RK

RK4



If closed curves then it's periodic

$$\left.\begin{aligned}\frac{df}{dt}&=0\\ \frac{dm}{dt}&=0\end{aligned}\right\}\text{fixed point}$$