Partial Differential Equations...but first...

Monday, 7 November 2022 13:03

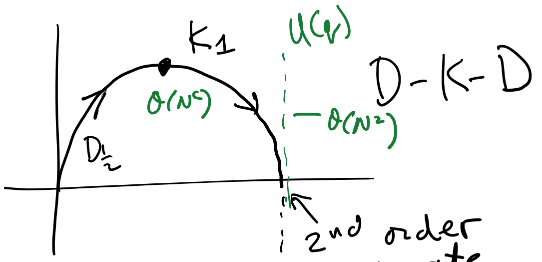
Variants of Leapfrog

$$D_{\frac{1}{2}} K_1 D_{\frac{1}{2}} \quad \left\{ \begin{matrix} p_1 \\ q_1 \\ p_2 \\ q_2 \\ \vdots \end{matrix} \right\}$$

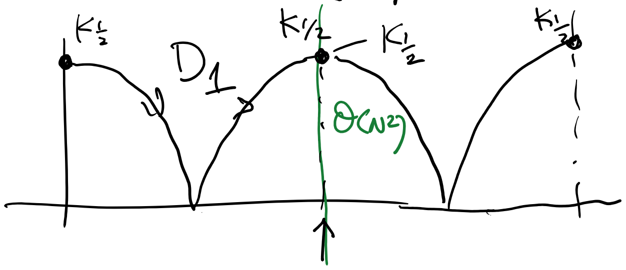
Umbrella diagrams

$$\boxed{K_{\frac{1}{2}} D_1 K_{\frac{1}{2}}}$$

~ Kick-Drift Kick



2nd order accurate $\{p,q\}$



$\{p,q\}$ are 2nd order accurate here!

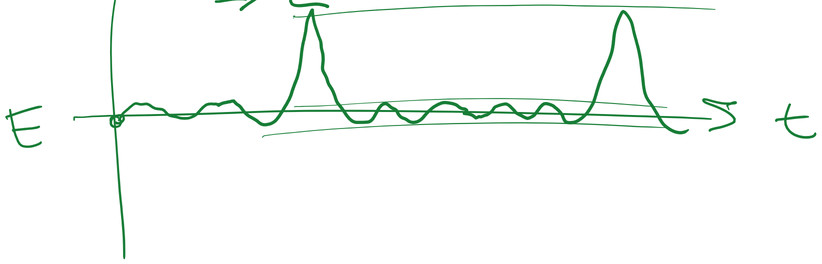
$\Rightarrow E$

$$E = T(p) + \underline{U(q)}$$

2nd-order accurate

same as difficulty as calc. the Force $O(N^2)$

$$H \neq \tilde{H}$$



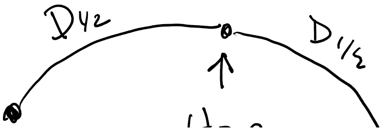
Seperability $\quad H = \underline{T(p)} + \underline{U(q)}$

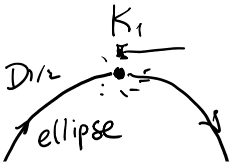
Not the only way for Planets

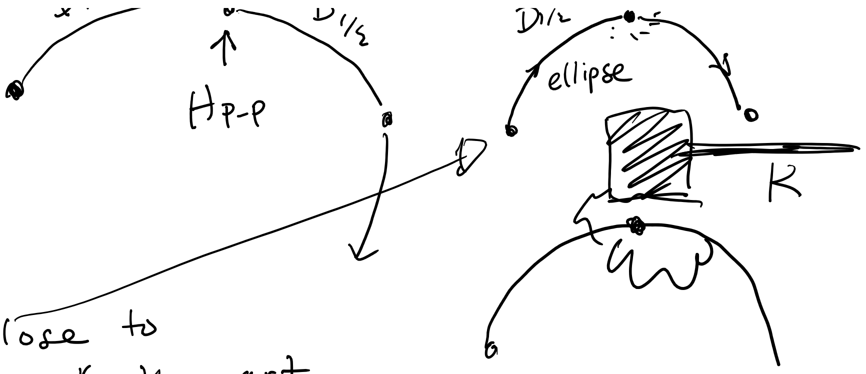
$$H = \underline{H_{P-S}} + \varepsilon H_{P-P}$$

2-body Problem Solve Kepler's Equation

1000 X more accurate!







is close to
State-of-the-art

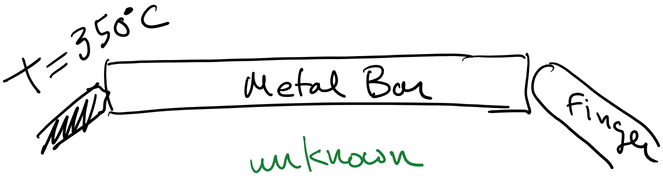
Partial Differential Equations

$$\boxed{\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}}$$

1-D Wave Equation
Hyperbolic P.D.E.
CLASS

$$\boxed{\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}}$$

1-D Diffusion Equation
Parabolic P.D.E.
CLASS



$$\frac{\partial^2 \overset{\text{unknown}}{u}}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \overset{\text{Given}}{\boxed{\rho(x,y)}}$$

2-D Poisson Equation

(CLASS: Elliptic P.D.E.s)

$$u(x,y) = ?$$

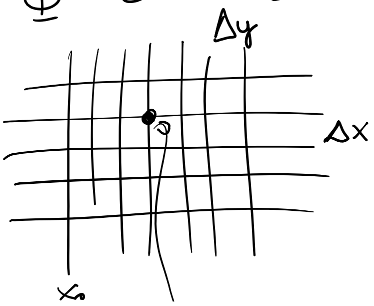
Laplace Equation $\rho(x,y)=0$

$$\nabla^2 u = 0$$ same

Electrostatic Potential in a Vacuum

$$\nabla^2 \underline{\Phi} = 0$$

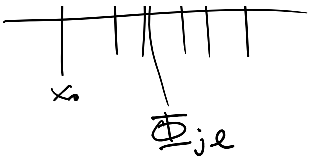
1st step: discretize the equation



Grid spacing $\Delta x, \Delta y$

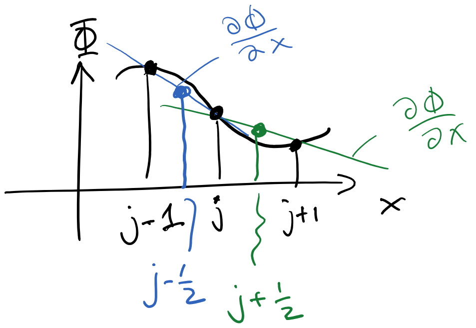
$$x_j = x_0 + j\Delta x \quad j=0,1\dots,J$$
$$y_\ell = y_0 + \ell\Delta y \quad \ell=0,1\dots L$$
$>N$

Simplify $\Delta = \Delta x = \Delta y$



Simplify $\Delta = \Delta x = \Delta y$

$$\frac{\partial^2 \Phi}{\partial x^2} = ?$$



$$\left.\frac{\partial \Phi}{\partial x}\right|_{j+\frac{1}{2}} \cong \frac{\phi_{j+1} - \phi_j}{\Delta}$$

$$\left.\frac{\partial \Phi}{\partial x}\right|_{j-\frac{1}{2}} \cong \frac{\phi_j - \phi_{j-1}}{\Delta}$$

$$\frac{\partial}{\partial x}\left[\frac{\partial \phi}{\partial x}\right] \cong \frac{\left.\frac{\partial \phi}{\partial x}\right|_{j+\frac{1}{2}} - \left.\frac{\partial \phi}{\partial x}\right|_{j-\frac{1}{2}}}{\Delta}$$

$$\left.\frac{\partial^2 \Phi}{\partial x^2}\right|_j \cong \frac{1}{\Delta^2}\left[\phi_{j+1} - 2\phi_j + \phi_{j-1}\right]$$

$$= \frac{1}{\Delta^2}\left[\begin{array}{ccc} \bullet & \bullet & \bullet \\ 1 & -2 & 1 \end{array}\right]$$

$$\left.\frac{\partial^2 \Phi}{\partial y^2}\right|_\ell = \frac{1}{\Delta^2}\left[\begin{array}{c} 1 \\ -2 \\ 1 \end{array}\right]$$

"Stencil"

$$\left[\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right]_{j,\ell} \cong \frac{1}{\Delta^2}\left[\begin{array}{ccc} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{array}\right]$$

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the equation $\nabla^2 \phi = 0$ Continuous

$$\frac{1}{\Delta^2}\left[\begin{array}{ccc} & \bullet & \\ \bullet & \bullet & \bullet \\ & \bullet & \end{array}\right]\phi = 0$$ Discrete

$$\frac{1}{\Delta^2}\left[\phi_{j+1,\ell} + \phi_{j-1,\ell} + \phi_{j,\ell+1} + \phi_{j,\ell-1} - 4\phi_{j,\ell}\right] = 0$$

$$\forall j,\ell : \phi_{j,\ell}^{(new)} = \frac{1}{4}\left[\phi_{j+1,\ell}^{(old)} + \phi_{j-1,\ell}^{(old)} + \phi_{j,\ell+1}^{(old)} + \phi_{j,\ell-1}^{(old)}\right]$$

iterate

Jacobi Method.

We have to iterate this until the change in ϕ_{je} over the grid is "small"

$$\phi_{je}^{(n+1)} = \phi_{je}^{(n)} + \underbrace{\frac{1}{4}\left(\phi_{j+1\ell}^{(n)} + \phi_{j-1,\ell}^{(n)} + \phi_{j\ell+1}^{(n)} + \phi_{j\ell-1}^{(n)} - 4\phi_{je}^{(n)}\right)}_{\text{correction}}$$

should get
smaller and
smaller!

Jacobi Method

$$N_{iter} \sim \frac{1}{2} p N^2$$

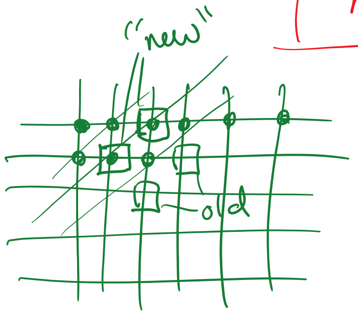
on an N×N grid to reduce the error by a factor of 10^{-p}.

Very slow

$$N_{ops} = N_{iter} \cdot N^2 \cdot \underbrace{N_{ops/point}}_{4}$$

$$\boxed{N_{ops} = 2pN^4}$$

$\mathcal{O}(N^4)$



Gauss Seidel
"Sweeps"

$\mathcal{O}(N^3)$: algorithm

$\mathcal{O}(N^2 \log N)$

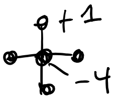
$\mathcal{O}(N^2)$ best

call Successive-Over-Relaxation (SOR)

Multigrid Method

"over do the correction somewhat"

$$\phi_{je}^{(n+1)} = \phi_{je}^{(n)} + \frac{\omega}{4}\underbrace{\left(\qquad \right)}_{\text{correction}}$$



$\omega = 1 \Rightarrow$ Jacobi

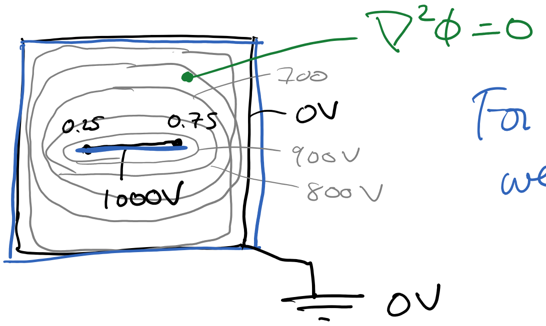
$\omega > 1 \Rightarrow$ SOR $(\omega < 2)$

If ω is optimal $\boxed{N_{iter} \sim \frac{1}{3} p N}$!

$$\omega \cong \frac{2}{1 + \pi/N}$$

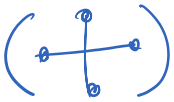
$\omega \rightarrow 1.7 - 1.9$

$\nabla^2 \phi = 0$



For blue points (BC) we want the correction to be equal to 0.

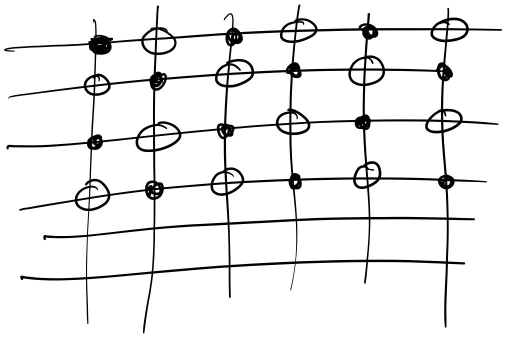
$$\phi_{je}^{(n+1)} = \phi_{je}^{(n)} + R_{je}\cdot\left(\quad\right)$$



0 if Boundary

$\frac{\omega}{4}$ if it is vaccuum

In Place (one grid of Φ)



Chess Board

1 - iteration

a) Sweep the Black points

b) Sweep the White points

Huge Parallel gains are possible....