What is an elliptic PDE

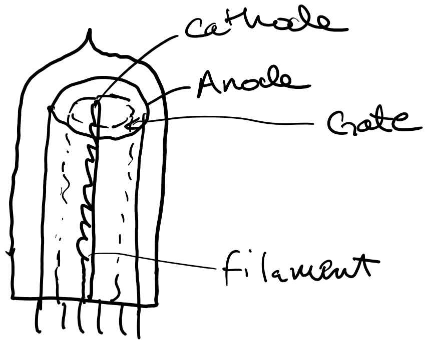
$$\underbrace{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \underline{\rho(x,y)}}_{\text{known}} \quad \text{2-D Poisson equation}$$

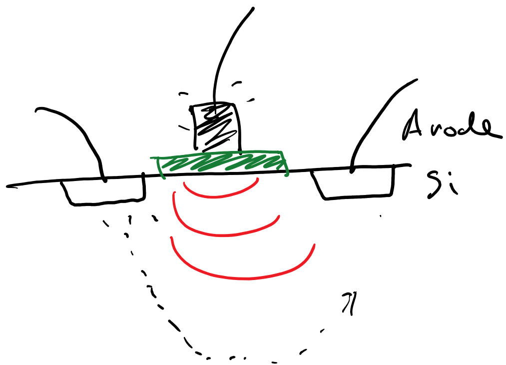
+ B.C.s → same $\nabla^2 u = \rho(x,y)$

Want to solve for u!

$\nabla^2 \phi = 0$ Laplace Equation

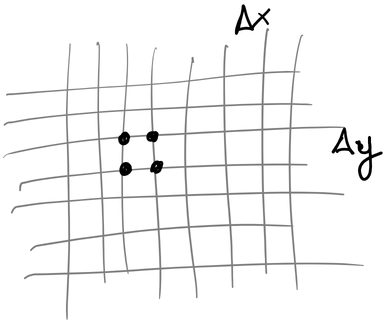
Examples





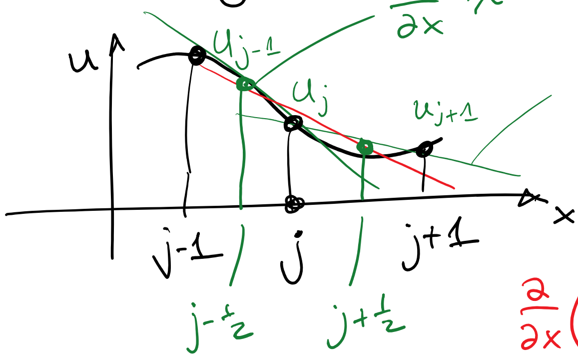
★ Discritize the continuous equation:

⟹ Write the operator as a grid operator and solve it algebraically.



$\Delta x = \Delta y = h$

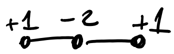
$$\frac{\partial u}{\partial x} \approx \frac{u_j - u_{j-1}}{h}$$



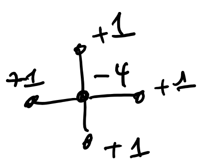
$$\frac{\partial u}{\partial x} \approx \frac{u_{j+1} - u_j}{h}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) \approx \frac{\left(\frac{\partial u}{\partial x}\right)_{j+\frac{1}{2}} - \left(\frac{\partial u}{\partial x}\right)_{j-\frac{1}{2}}}{h}$$

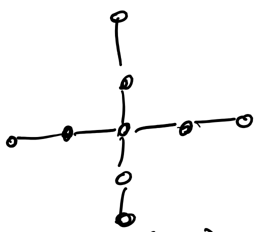
$$\left.\frac{\partial^2 u}{\partial x^2}\right|_j \approx \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}$$



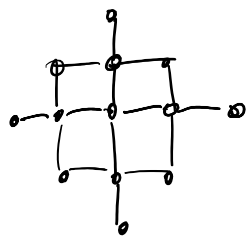
In 2-D



$O(h^2)$



$O(h^4)$



$O(h^4)$

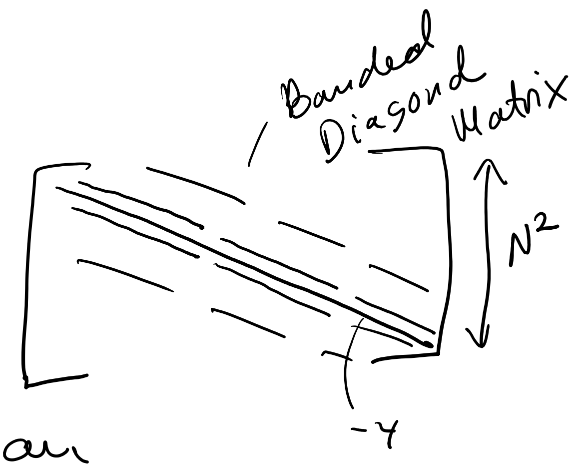
$$\nabla^2 \phi = 0$$

$$[\text{stencil}]\,\phi = 0 \implies \frac{1}{h^2}\left[\phi_{j+1,l} + \phi_{j-1,l} + \phi_{j,l+1} + \phi_{j,l-1} - 4\phi_{jl}\right] = 0$$

$$\underbrace{\phi_{jl}}_{N^2 \text{ elements}} \longrightarrow \underbrace{\phi_m}_{\text{also } N^2 \text{ elements}} \qquad m \in [1 \ldots N^2]$$

ϕ_m is a big vector.

$$\underline{\underline{M}}\,\underline{\phi} = \underline{b} \qquad M =$$



So we have an $\underline{\text{exact}}$ solution to the $\underline{\text{discrete}}$ problem, (at least in principle)

ϕ_h - exact solution

$\tilde{\phi}_h$ - some approximate solution

Jacobi Method

$$\tilde{\phi}^{new}_{jl} \approx \frac{1}{4}\left[\tilde{\phi}^{old}_{j+1,l} + \tilde{\phi}^{old}_{j-1,l} + \tilde{\phi}^{old}_{j,l+1} + \tilde{\phi}^{old}_{j,l-1}\right]$$

old = new

$$\tilde{\phi}^{new}_{jl} - \tilde{\phi}^{OLD}_{jl} \approx \frac{1}{4}\left[\text{stencil}\right]\phi$$

+1

−4

no $\frac{1}{h^2}$ in these stencils?

$\tilde{\phi}$ $\approx$ $\frac{1}{4}$ [ ] Residual

$$\tilde{R}_{je} \approx \frac{1}{4}\begin{bmatrix} & \circ & \\ \circ & + & \circ \\ & \circ & \end{bmatrix} \quad \text{Residual}$$

$N_{iter} \sim \frac{1}{2} p N^2$ on an $N \times N$ grid to reduce the error by a factor of 10^{-p}

Number of operations $\sim \mathcal{O}(N^4)$

$$\tilde{\Phi}^{new}_{je} - \tilde{\Phi}^{OLD}_{je} = \omega \tilde{R}_{je} \qquad \boxed{0 \le \omega < 2}$$

SOR $\mathcal{O}(N^3)$

MULTIGRID $\mathcal{O}(N^2)$