SINS 6: Population Growth, Chaos and Fractals

21 October 2019 12:08

Constant Growth rate r for the population:

$$\frac{P_{n+1} - P_n}{P_n} =: r$$

or $P_{n+1} = (1+r)P_n$

There is solution for this for all n an r,

$$P_{n+1} = (1+r)^n P_0$$

Population explosion!

$$\frac{1}{P}\frac{dP}{dt} = r$$

$$\frac{d \ln P}{dt} = r$$

$$\int_{\ln P_0}^{\ln P} d\ln P = \int_0^t r\,dt$$

$$\ln \frac{P}{P_0} = rt$$

$$P = P_0 e^{rt}$$

Normalize the population with respect to some "maximal" size, N.

$$p = P/N$$

p	r ← not constant!
1	0
small	large and positive
~ 1	small
>1	negative

Verhulst

$r \propto (1-p_n)$

$= k(1-p_n)$

$$\frac{p_{n+1} - p_n}{p_n} = k(1-p_n)$$

$$\frac{dP}{dt} = rP(1-P)$$

$$p_{n+1} = p_n + k\,p_n(1-p_n)$$

Quadratic Dependence

Non-Linear System!

Even in this very simple non-linear system, there is no general closed form solution.

But it is deterministic

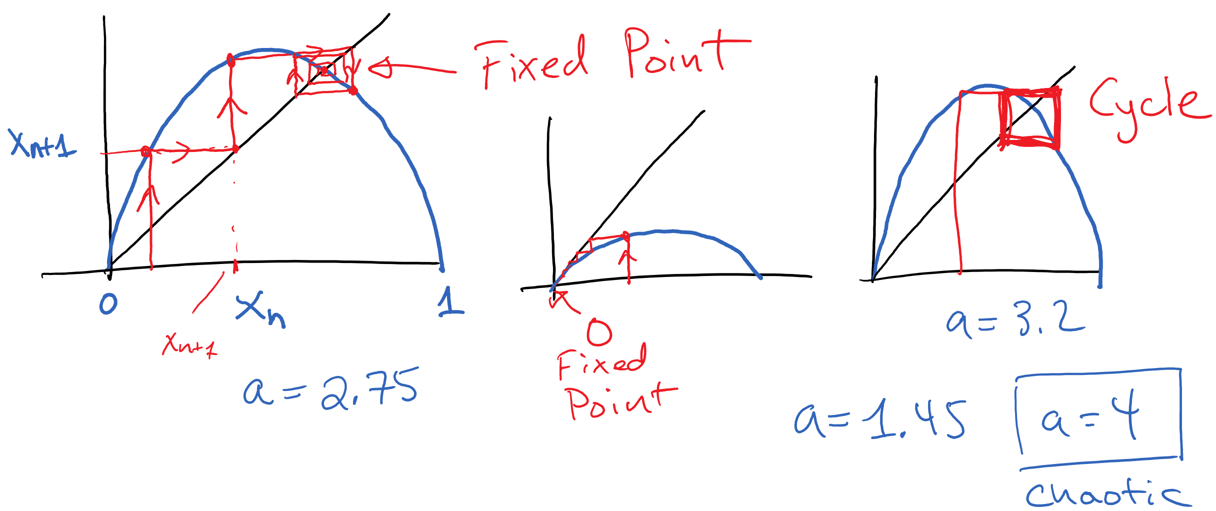
Eg, $r = 3$ $\quad p_0 = 0.01$

$p_0' = 0.00999999999\ldots$

$$x_{n+1} = a\,x_n(1-x_n)$$ Logistic Equation

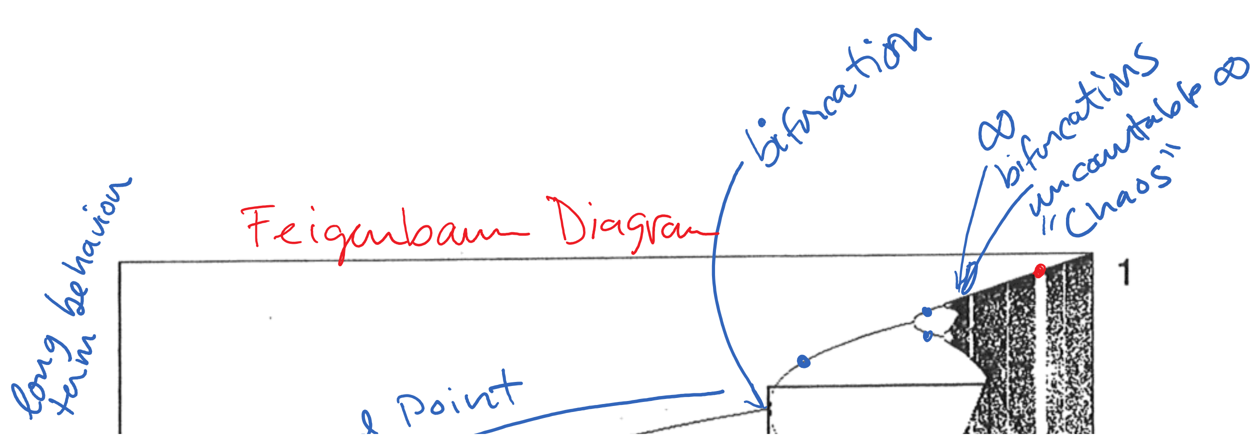
$$x \in [0,1]$$

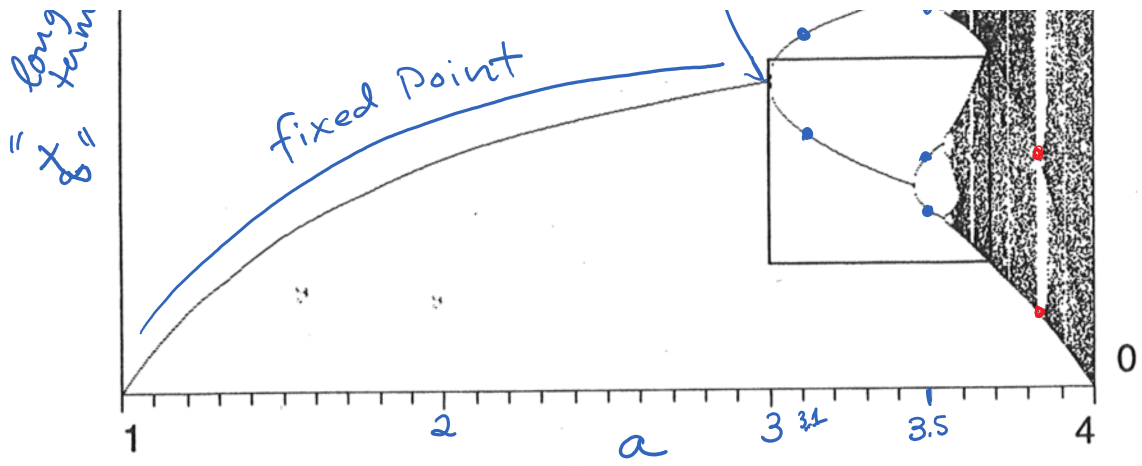
$$a \in [0,4]$$

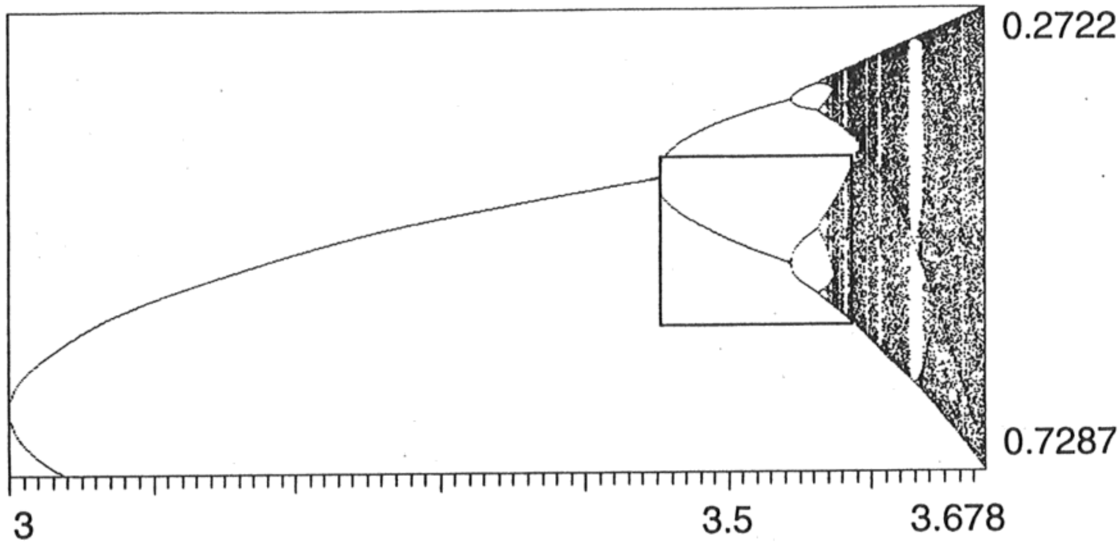


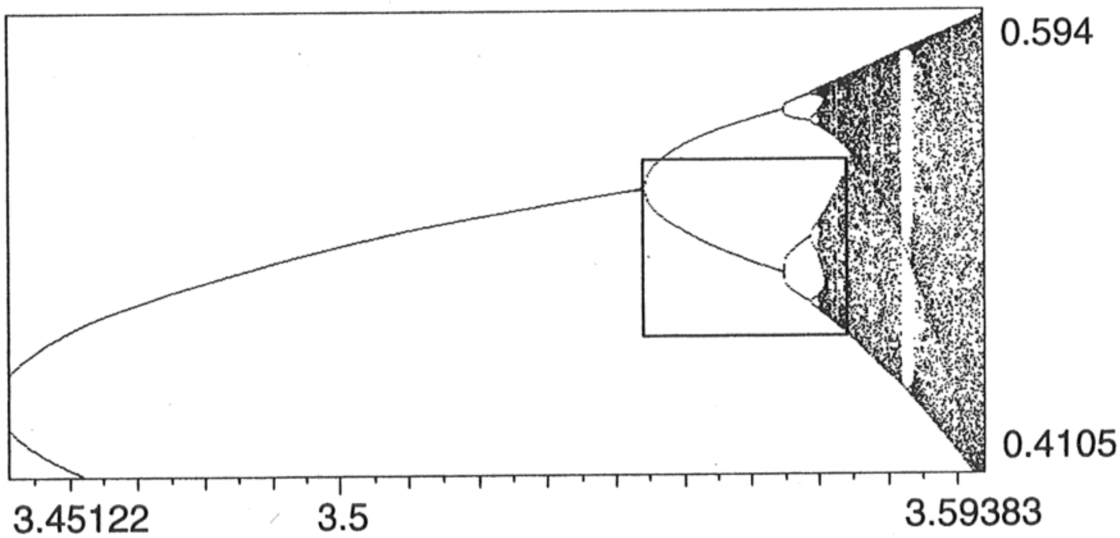
Feigenbaum Point 3.5699456…

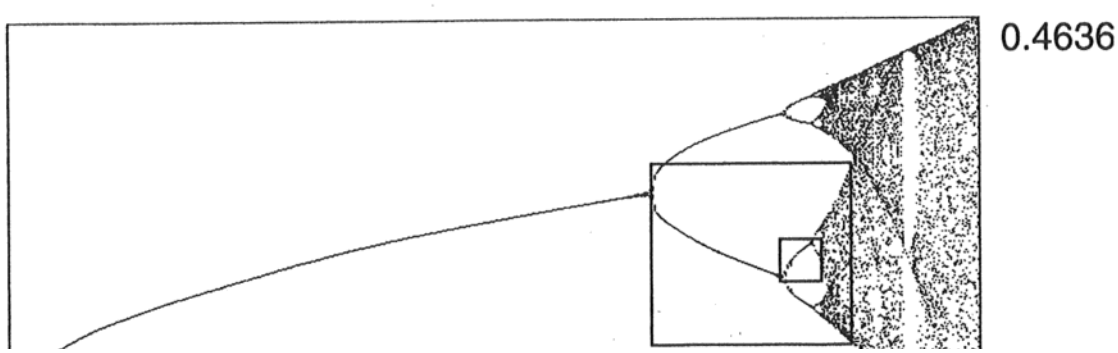
Feigenbaum Diagram

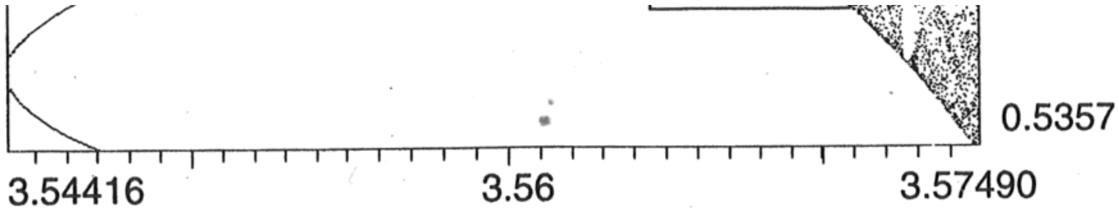



long term
" x_∞ "
fixed Point
0
1
2
a
3
3.1
3.5
4

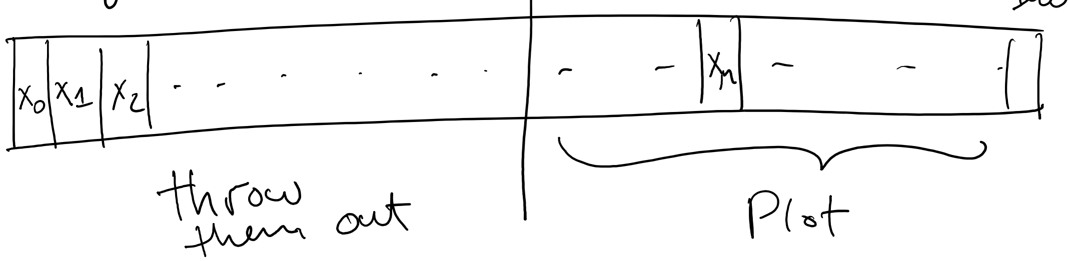

0.2722
0.7287
3
3.5
3.678

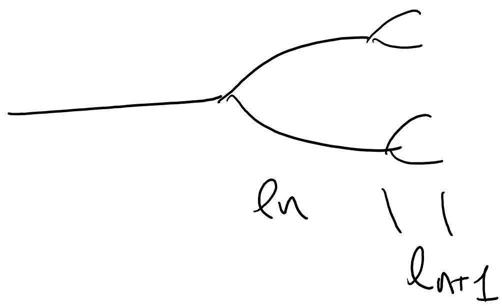

0.594
0.4105
3.45122
3.5
3.59383


0.4636



Array for a given a





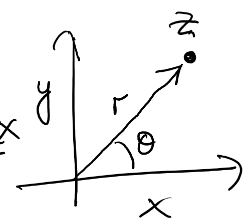
Universal constant!

$$\frac{l_n}{l_{n+1}} = \text{Feigenbaum Constant}$$

as $n \to \infty$ is given by 4.6692...

Complex Iterators: $i^2 = -1$ $\quad z = (x + iy)$

Real part $\quad$ Complex part



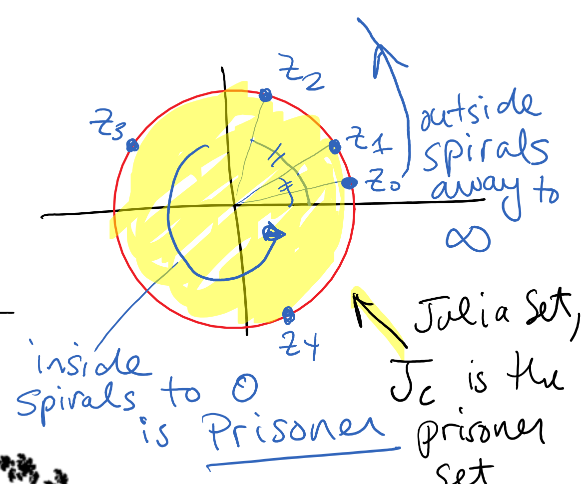
let $z = x + iy$
$w = u + iv$

$z = re^{i\theta}$

$$z \cdot w = (xu + iyu + ixv + \overset{-1}{i^2} yv)$$
$$= (xu - yv) + i(yu + xv)$$
$$z^2 = (re^{i\theta})^2 = r^2 e^{i(2\theta)}$$

Consider $z_{n+1} = z_n^2$ $\Longrightarrow$

Non-Linear



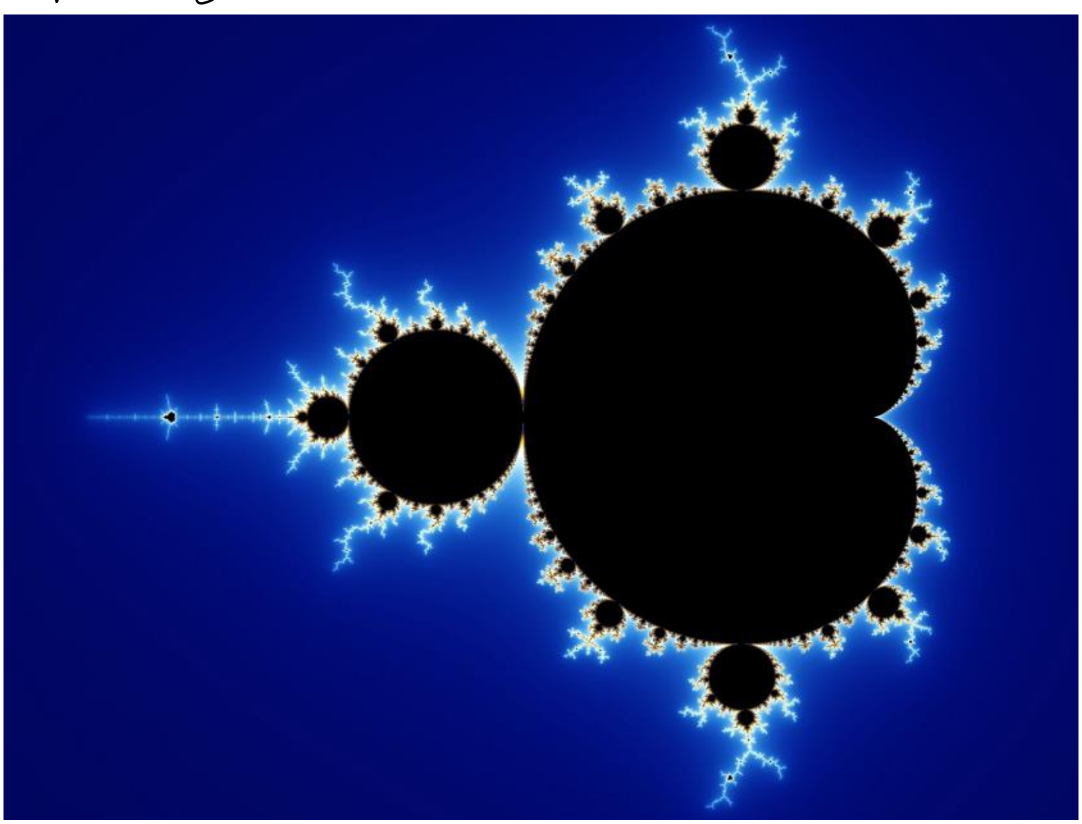
Now $\boxed{z_{n+1} = z_n^2 + C}$ Iterator.
What is the Julia Set?

E.g. for $c = -0.5 + 0.5i$
A connected J_c!

For some c we get an unconnected J_c!

The Mandelbrot Set: Set of all c for which the Julia set is a connected set!

$$M = \{ c \in \mathbb{C} \mid J_c \text{ is connected} \}$$



Another definition:

$$M = \{ z_0 = c \in \mathbb{C} \mid z_{n+1} = z_n^2 + c < \infty \}$$

easier to program!

If $|z_n| > r(c)$ a critical radius, then it will always escape to infinity!

$$r(c) := \max(|c|, 2)$$

Colors: according to how many steps to escape (how close to being a part of the Mandelbrot Set).

Do forall c

```
k = 0
z = c
while (k < 100) {
    if (|z| > r(c)) {
        Draw point with Color(k);
        return (c ∉ M);
    }
    z = z*z + c        z^2 Complex
    k = k + 1
}
Draw point with Color("black");
return (c ∈ M);
```