

Flux = material flowing through the surface per unit time.

$$\rho \Delta x \cdot \delta y \delta z / \Delta t = \rho \underline{u} \cdot \delta x \delta y$$

$$\rho \underline{u} \text{ at } x + \delta x \text{ is } \approx \rho \underline{u} + \frac{1}{2} \delta x \frac{\partial(\rho \underline{u})}{\partial x} \Big|_{x,t}$$

$$\left\{ \rho \underline{u} + \frac{1}{2} \delta x \frac{\partial(\rho \underline{u})}{\partial x} \Big|_{x,t} \right\} \delta y \delta z$$

Flux through the "front" surface

$$\left\{ \rho \underline{u} - \frac{1}{2} \delta x \frac{\partial(\rho \underline{u})}{\partial x} \Big|_{x,t} \right\} \delta y \delta z$$

to simplify let \underline{u} in x direction only.

Flux into the cube is then given by the difference of the above:

$$- \frac{1}{2} \delta x \frac{\partial(\rho \underline{u})}{\partial x} \Big|_{x,t} \delta y \delta z = - \frac{\partial(\rho \underline{u})}{\partial x} \delta V$$

$$\boxed{- \frac{\partial(\rho \underline{u})}{\partial x} \delta V = \frac{\partial \rho}{\partial t} \delta V}$$

$$\frac{\partial \rho}{\partial t} = - \frac{\partial(\rho \underline{u})}{\partial x} \quad \text{for flow in } x \text{ direction.}$$

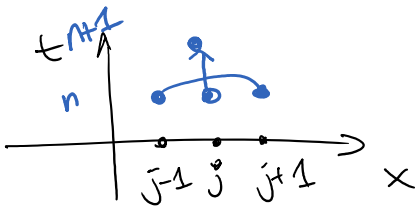
$$\underline{\dot{\rho}} = - \nabla \cdot (\rho \underline{u}) \quad \text{or} \quad \boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0}$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \underline{u}) \Rightarrow \left. \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0 \right\} \text{Conservation of Mass.}$$

+ Conservation of Momentum of (3 equations)

+ Conservation of Energy (1 equation) for later ...

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} = 0 \quad \text{"Linear Advection Equation"}$$



$$\frac{\partial \rho}{\partial x} \Big|_j \approx \frac{\rho_{j+1}^{(n)} - \rho_{j-1}^{(n)}}{2\Delta x}$$

$$\frac{\partial \rho}{\partial t} \approx \frac{\rho_j^{(n+1)} - \rho_j^{(n)}}{\Delta t}$$

$$\rho_j^{(n+1)} = \rho_j^{(n)} - \frac{1}{2}c(\rho_{j+1}^{(n)} - \rho_{j-1}^{(n)})$$

$$\text{Courant Number } c = \frac{\Delta t u}{\Delta x}$$

von Neumann Stability Analysis

$$\rho_j^{(n)} = A^n e^{ij\theta} \quad i = \sqrt{-1}$$

$$\cancel{A^{n+1} e^{ij\theta}} = \cancel{A^n e^{ij\theta}} - \frac{1}{2}c \cancel{A^n} (e^{i(j+1)\theta} - e^{i(j-1)\theta})$$

cancel $e^{ij\theta}$

$$A = 1 - c \left(\frac{e^{i\theta} - e^{-i\theta}}{2} \right)$$

$$\boxed{A = 1 - ic \sin(\theta)}$$

$$\begin{aligned} A^* A &= \|A\|^2 \\ &= 1 + c^2 \sin^2 \theta > 1!! \end{aligned}$$

Completely useless

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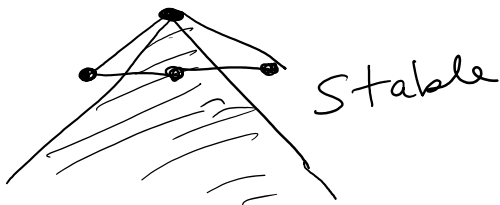
LAX Method: $\rho_j^{(n)} \rightarrow \frac{1}{2}(\rho_{j+1}^{(n)} + \rho_{j-1}^{(n)})$

$$\rho_j^{(n+1)} = \frac{1}{2}(\rho_{j+1}^{(n)} + \rho_{j-1}^{(n)}) + \frac{1}{2}c(\rho_{j+1}^{(n)} - \rho_{j-1}^{(n)})$$

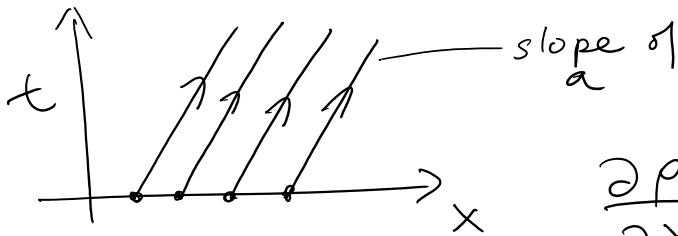
$$\Rightarrow A = \cos \Theta - ic \sin \Theta$$

Stable when $|c| \leq 1$ or $\frac{|a| \Delta t}{\Delta x} < 1$

Courant Condition
or CFL Condition



$a < \frac{\Delta x}{\Delta t}$ Information must be able to flow on the grid faster than in "reality".



$$\frac{\partial \rho}{\partial x} \approx \frac{\rho_j^{(n)} - \rho_{j-1}^{(n)}}{\Delta x} \quad \text{when } a > 0$$



$$\frac{\rho_{j+1}^{(n)} - \rho_j^{(n)}}{\Delta x} \quad \text{when } a < 0$$

CFL: $\|A\| < 1$ when $0 \leq c < 1$

Upwind Schemes

$$\rho_j^{(n+1)} = \rho_j^{(n)} - c(\rho_j^{(n)} - \rho_{j-1}^{(n)}) \quad \text{for } a > 0$$

M_j N $1 \leq j \leq N-1$

LAX-WENDROFF Method:

$$\rho_j^{(n+1)} = \frac{1}{2} c(1+c) \rho_{j-1}^{(n)} + (1-c^2) \rho_j^{(n)} - \frac{1}{2} c(1-c) \rho_{j+1}^{(n)}$$

