SPH (main numerical equations)

Monday, 21 March 2022 12:44

We used $\underline{u}$ for fluid velocity, but I want to change to use $\underline{v}$ instead → a new field for every particle.

$$\nabla \cdot \underline{v} \cong \sum_b \frac{m_b}{\rho_b} \underline{v}_b \cdot \nabla W(|\underline{r}-\underline{r}_b|, h)$$

Monahan cubic spline

size of the Kernel

$$\nabla \cdot \underline{v} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(v_x \hat{i} + v_y \hat{j} + v_z \hat{k}\right)$$

$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$\nabla \cdot \underline{v} = \frac{1}{\rho}\left[\nabla \cdot (\rho \underline{v}) - \underline{v} \cdot \nabla \rho\right]$$

$$\nabla \cdot \underline{v}_a \cong \frac{1}{\rho_a}\left[\sum_b m_b \frac{\cancel{\rho_b} \underline{v}_b}{\cancel{\rho_b}} \cdot \nabla_a W_{ab} - \underline{v}_a \cdot \sum_b m_b \nabla_a W_{ab}\right]$$

$$\rho_a \nabla \cdot \underline{v}_a \cong \sum_b m_b (\underline{v}_b - \underline{v}_a) \cdot \nabla_a W_{ab}$$

$(W(|\underline{r}_a - \underline{r}_b|, h)$

switch $a \Leftrightarrow b$

$$\frac{de}{dt} = -\frac{P}{\rho} \nabla \cdot \underline{v}$$

energy
$$\boxed{\frac{de_a}{dt} = \left(\frac{P_a}{\rho_a^2}\right) \sum_b m_b (\underline{v}_a - \underline{v}_b) \cdot \nabla_a W_{ab}}$$

Benz Formulation

continuity
$$\rho_a = \sum_b m_b W_{ab}$$

momentum equation
$$\frac{d\underline{v}_a}{dt} = -\frac{\rho_a \nabla P_a}{\rho_a^2}$$

$$\frac{d\underline{v}_a}{dt} = -\frac{1}{\rho_a^2} \sum_b m_b (P_b - P_a) \nabla_a W_{ab}$$

Force = 0 for constant Pressure, but it does not conserve momentum

$F_{ab} = -F_{ba}$

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Force $= 0$ for constant pressure, but it does not conserve momentum nor angular momentum.

Instead: $\dfrac{\nabla P}{\rho} = \nabla\left(\dfrac{P}{\rho}\right) + \dfrac{P}{\rho^2}\nabla\rho$

$$\frac{d\underline{v}_a}{dt} = -\sum_b m_b \left(\frac{P_a}{\rho_a^2} + \frac{P_b}{\rho_b^2}\right) \nabla_a W_{ab}(r_{ab}, h)$$

same for a & b?

$a \Leftrightarrow b \qquad \nabla_a \rightarrow -\nabla_b$

Newton's 3rd Law $\quad F_{ab} = -F_{ba}$

How to conserve momentum when particles have different kernel sizes h_a, h_b?

Symmetrizing the Kernel

$a \longleftrightarrow b \qquad \left(\dfrac{P_a}{\rho_a^2} + \dfrac{P_b}{\rho_b^2}\right)$ ✓

$W_{ab}(r_{ab}, \underline{h_a})$ $\leftarrow$ $h_{ab} = \frac{1}{2}(h_a + h_b)$ ✓

nice, but difficult to implement

$$W_{ab} = \frac{1}{2}\Big(W(r_{ab}, h_a) + W(r_{ab}, h_b) \Big) \quad ✓$$

"easy" to implement.

① $\dfrac{d\underline{v}_a}{dt} \mathrel{+}= -\dfrac{1}{2} \boxed{\sum_b} m_b \underbrace{F_{ab}}_{\left(\frac{P_a}{\quad} \quad \rho_b^2\right)} \boxed{\nabla_a W(r_{ab}, h_a)}$ "gather"

② $\dfrac{d\underline{v}_a}{dt} \mathrel{+}= -\dfrac{1}{2} \sum_b m_b F_{ab} \boxed{\nabla_a W(r_{ab}, h_b)}$ "gather"

rewrite ② swapping $a \Leftrightarrow b$

$$\frac{d\underline{v}_b}{dt} \mathrel{+}= -\frac{1}{2}\sum m_a F_{ab} \nabla_b W(r_{ab}, h_a)$$

$$\frac{dv_b}{dt} += -\frac{1}{2}\sum_a m_a F_{ab} \nabla_b W(r_{ab}, h_a)$$

but $\nabla_b = -\nabla_a$

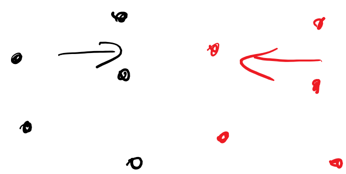
$$\frac{dv_b}{dt} += \frac{1}{2}\boxed{\sum_a} m_a \boxed{F_{ab} \nabla_a W(r_{ab}, h_a)}$$

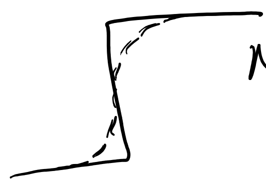
"Scatter"

Only need to calculate the contribution in $\boxed{}$ once for each neighbor

Gather scatter algorithm.

Remember to initialize $\frac{dv_a}{dt} = 0$ for all particles.





Shock

Navier-Stokes Equations

"Direct" Numerical

Add Artificial Viscosity

$$F_{ab} = \frac{P_a}{\rho_a^2} + \frac{P_b}{\rho_b^2} + \Pi_{ab}$$

$$\Pi_{ab} = \begin{cases} \dfrac{-\alpha \bar{c}_{ab} \mu_{ab} + \beta \mu_{ab}^2}{\bar{\rho}_{ab}}, & v_{ab} \cdot r_{ab} < 0 \\ 0, & v_{ab} \cdot r_{ab} \geq 0 \end{cases}$$

$\bar{c}_{ab} = \frac{1}{2}(c_a + c_b)$

$\bar{\rho}_{ab} = \frac{1}{2}(\rho_a + \rho_b)$

$\beta = 2\alpha$

$\alpha = 1 \quad \beta = 2$

$\alpha = 0.5 \quad \beta = 1$

$$\mu_{ab} = \frac{\bar{h}_{ab}\, v_{ab} \cdot r_{ab}}{r_{ab}^2 + \eta^2}$$

small number

$\bar{h}_{ab} = \frac{1}{2}(h_a + h_b)$

... reasonable

to keep a reasonable denominator

$$e_a = \frac{P_a}{\rho_a(\gamma-1)} \qquad c_a = \sqrt{\gamma(\gamma-1)e_a}$$

P this way $\gamma\rho$ Need c

Let's write the code ...

Variables for particles: $\underline{r}, \underline{v}, e, c, \rho, h$

Need to calculate: $\underline{a} = \frac{d\underline{v}}{dt}$ $\quad \dot{e} = \frac{de}{dt}$

☆ Need 2 extra temporary variables → ODE integration using a Leapfrog-like algorithm.

$\underline{v}_{pred}$, e_{pred}

Functions: DRIFT1(), DRIFT2()
KICK(), CALCFORCES()

SPH:

```
DRIFT1(Δt=0)      Bootstrap step for v_pred, e_pred
CALCFORCE()
for (step=0; step<NSTEPS; ++step) {
    DRIFT1(Δt/2)
    CALCFORCE()
    KICK(Δt)
    DRIFT2(Δt/2)
}
```

CALCFORCE:

TREEBUILD

NN-DENSITY ← all particles calculate ρ_a

CALCSOUND ← all particles calculate c

NN-SPHFORCE ← All calculate $\underline{a}, \dot{e}$

TREE walks

DRIFT1(Δt):

$$\underline{r} \mathrel{+}= \underline{v}\,\Delta t$$

$$\underline{v}_{pred} = \underline{v} + \underline{a}\,\Delta t$$

$$e_{pred} = e + \dot{e}\,\Delta t$$

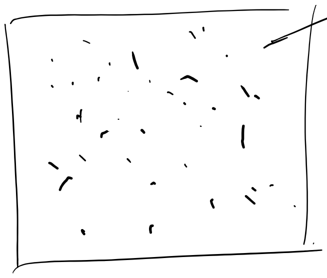
DRIFT2(Δt):

$$\underline{r} \mathrel{+}= \underline{v}\,\Delta t$$

KICK(Δt):

$$\underline{v} \mathrel{+}= \underline{a}\,\Delta t$$

$$\underline{e} \mathrel{+}= \dot{e}\,\Delta t$$



$v = 0$

$e = ?$

Periodic