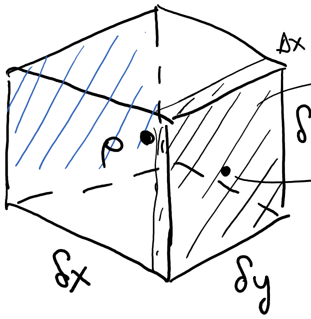
Hyperbolic PDEs

23 November 2020 12:53

$\delta \to$ very small



Area : $\delta y \delta z$

density at face center

$$\rho(x+\tfrac{1}{2}\delta x, t) \approx \rho(x,t) + \tfrac{1}{2}\delta x \left.\frac{\partial \rho}{\partial x}\right|_{x,t}$$

Flux = material flowing through the surface per unit time.

$$\rho \Delta x \cdot \delta y \delta z / \Delta t = \rho u \,\delta y \delta z$$

u velocity in the x-direction

ρu at $x+\frac{1}{2}\delta x$ is $\approx \rho u + \frac{1}{2}\delta x \left.\frac{\partial(\rho u)}{\partial x}\right|_{x,t}$

Flux through the "front" surface is then given by:

$$\left\{\rho u + \frac{1}{2}\delta x \left.\frac{\partial(\rho u)}{\partial x}\right|_{x,t}\right\} \delta y \delta z$$

Flux through the "back" surface is given by:

$$\left\{\rho u - \frac{1}{2}\delta x \left.\frac{\partial(\rho u)}{\partial x}\right|_{x,t}\right\} \delta y \delta z$$

Flux into the cube is then given by the difference of the above 2 expressions.

$$-1\,\delta x \left.\frac{\partial(\rho u)}{\partial x}\right|_{x,t} \delta y \delta z = \boxed{-\frac{\partial(\rho u)}{\partial x}\delta V}$$

$$\boxed{\frac{\partial \rho}{\partial t}\cdot \delta V}$$

this change in mass in the cube is the same, as long as mass is conserved!

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho u)}{\partial x}$$ – for flow only in the x-direction.

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \underline{u})$$

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0}$$

Conservation of Mass.
(hyperbolic PDE!)

For Later...

+ Conservation of Momentum (3 in 3-D)
"F = ma"

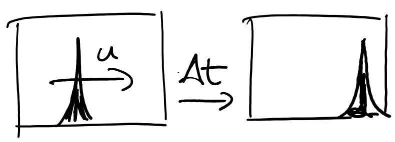
+ Conservation of Energy

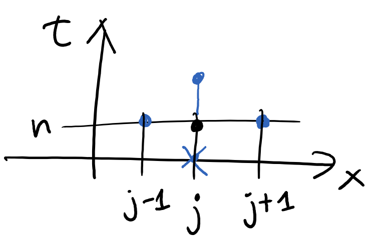
Hydrodynamics

Linear Advection Equation

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} = 0$$

How to treat this numerically.





$$\left.\frac{\partial \rho}{\partial x}\right|_j^{(n)} \approx \frac{\rho_{j+1}^{(n)} - \rho_{j-1}^{(n)}}{2\Delta x}$$

$$\left.\frac{\partial \rho}{\partial t}\right|_j \approx \frac{\rho_j^{(n+1)} - \rho_j^{(n)}}{\Delta t}$$

$$\boxed{\rho_j^{(n+1)} = \boxed{\rho_j^{(n)}} - \frac{1}{2} c \left(\rho_{j+1}^{(n)} - \rho_{j-1}^{(n)}\right)}$$

Proposal
Any good?

$$c = \frac{\Delta t \, u}{\Delta x}$$

Courant Number

Von Neumann Stability Analysis:

$$\rho_j^{(n)} = A^n e^{ij\theta} \qquad i \equiv \sqrt{-1}$$

j is a grid index!

$$A^{n+1} \cancel{e^{ij\theta}} = A^n \cancel{e^{ij\theta}} - \frac{1}{2} c A^n \left(e^{i(j+1)\theta} - e^{i(j-1)\theta}\right)$$

Cancel $A^n e^{ij\theta}$

$$A = 1 - c \left(\underline{e^{i\theta} - e^{-i\theta}}\right)$$

$$A = 1 - c\left(\frac{e^{i\theta} - e^{-i\theta}}{2}\right)$$

$$\boxed{A = 1 - iC\sin\theta}$$

$$A^*A \equiv |A^2| = 1 + c^2\sin^2\theta > 1 !!$$

Always unstable

Completely Useless.

LAX Method: $\rho_j^{(n)} \longrightarrow \frac{1}{2}\left(\rho_{j+1}^{(n)} + \rho_{j-1}^{(n)}\right)$

$$\rho_j^{(n+1)} = \frac{1}{2}\left(\rho_{j+1}^{(n)} + \rho_{j-1}^{(n)}\right) - \frac{1}{2}c\left(\rho_{j+1}^{(n)} - \rho_{j-1}^{(n)}\right)$$

$$\Rightarrow \boxed{A = \cos\theta - iC\sin\theta}$$

Stable when $|c| \leq 1$

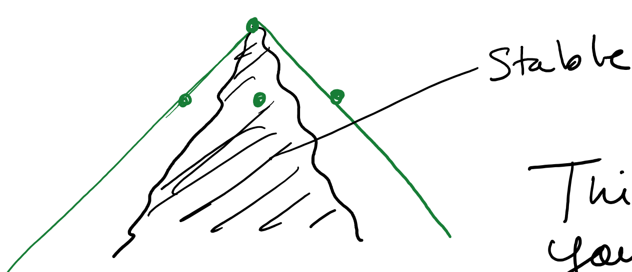
or $\dfrac{|u|\,\Delta t}{\Delta x} < 1$

Courant Condition or CFL Condition.

$$|u| < \frac{\Delta x}{\Delta t}$$

Physical Velocity — Grid Velocity

Physical information must not travel faster than the maximum velocity on the grid.



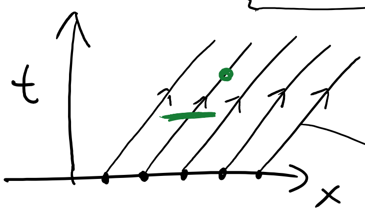
This is also the best you can do; you always have a CFL condition in

hyperbolic Systems.

Upwind Scheme:

$$\boxed{\rho_j^{(n+1)} = \rho_j^{(n)} - c\left(\rho_j^{(n)} - \rho_{j-1}^{(n)}\right) \quad \text{for } u>0}$$

$$- c\left(\rho_{j+1}^{(n)} - \rho_j^{(n)}\right) \quad \text{for } u<0$$



Slope is u

Characteristics: "Lines along which information flows".

Lax-WENDroff Method:

$$\rho_j^{(n+1)} = \frac{1}{2}c(1+c)\rho_{j-1}^{(n)} + (1-c^2)\rho_j^{(n)} - \frac{1}{2}c(1-c)\rho_{j+1}^{(n)}$$

for $u>0$

Fixed (sorry)