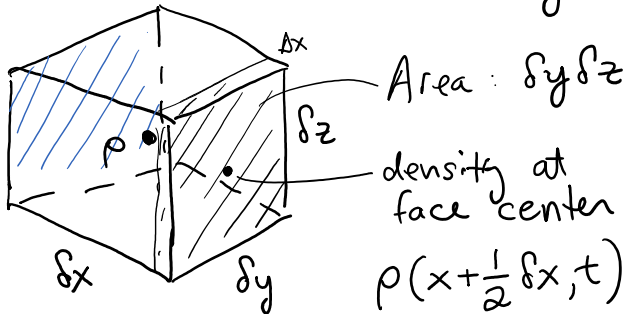


$\delta \rightarrow$  very small

$$\rho(x + \frac{1}{2} \delta x, t) \approx \rho(x, t) + \frac{1}{2} \delta x \left. \frac{\partial \rho}{\partial x} \right|_{x, t}$$

Flux = material flowing through the surface per unit time.

$$\rho \Delta x \cdot \delta y \delta z / \Delta t = \rho u \delta y \delta z$$

$u$  velocity in the x-direction

$$\rho u \text{ at } x + \frac{1}{2} \delta x \text{ is } \approx \rho u + \frac{1}{2} \delta x \left. \frac{\partial(\rho u)}{\partial x} \right|_{x, t}$$

Flux through the "front" surface is then given by:

$$\left\{ \rho u + \frac{1}{2} \delta x \left. \frac{\partial(\rho u)}{\partial x} \right|_{x, t} \right\} \delta y \delta z$$

Flux through the "back" surface is given by:

$$\left\{ \rho u - \frac{1}{2} \delta x \left. \frac{\partial(\rho u)}{\partial x} \right|_{x, t} \right\} \delta y \delta z$$

Flux into the cube is then given by the difference of the above 2 expressions.

$$- \frac{1}{2} \delta x \left. \frac{\partial(\rho u)}{\partial x} \right|_{x, t} \delta y \delta z = \boxed{- \frac{\partial(\rho u)}{\partial x} \delta V}$$

$$\boxed{\frac{\partial \rho}{\partial t} \cdot \delta V}$$

this change in mass in the cube is the same, as long as mass is conserved!

$$\frac{\partial \rho}{\partial t} = - \frac{\partial(\rho u)}{\partial x} \quad \text{— for flow only in the x-direction.}$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \underline{u})$$

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0}$$

Conservation of Mass.  
(hyperbolic PDE!)

For Later...

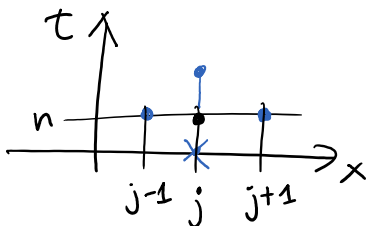
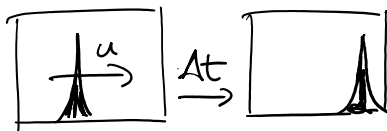
+ Conservation of Momentum ("F=ma") (3 in 3-D)

+ Conservation of Energy Hydrodynamics

### Linear Advection Equation

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} = 0$$

How to treat this numerically.



$$\left. \frac{\partial \rho}{\partial x} \right|_j^{(n)} \approx \frac{\rho_{j+1}^{(n)} - \rho_{j-1}^{(n)}}{2\Delta x}$$

$$\left. \frac{\partial \rho}{\partial t} \right|_j \approx \frac{\rho_j^{(n+1)} - \rho_j^{(n)}}{\Delta t}$$

$$\boxed{\rho_j^{(n+1)} = \rho_j^{(n)} - \frac{1}{2} c (\rho_{j+1}^{(n)} - \rho_{j-1}^{(n)})}$$

Proposal  
Any good?

$$c = \frac{\Delta t u}{\Delta x} \quad \text{Courant Number}$$

van Neumann Stability Analysis:

$$\rho_j^{(n)} = A^n e^{ij\theta} \quad i = \sqrt{-1}$$

j is a grid index!

$$A^{n+1} e^{ij\theta} = A^n e^{ij\theta} - \frac{1}{2} c A^n (e^{i(j+1)\theta} - e^{i(j-1)\theta})$$

Cancel  $A^n e^{ij\theta}$

$$A = 1 - c \frac{(e^{i\theta} - e^{-i\theta})}{2}$$

$$A = 1 - c \left( \frac{e^{i\theta} - e^{-i\theta}}{2} \right)$$

$$\boxed{A = 1 - iC \sin \theta}$$

$$A^*A \equiv |A|^2 = 1 + c^2 \sin^2 \theta > 1!!$$

Always unstable

Completely Useless.

LAX Method:  $\rho_j^{(n)} \rightarrow \frac{1}{2} (\rho_{j+1}^{(n)} + \rho_{j-1}^{(n)})$

$$\rho_j^{(n+1)} = \frac{1}{2} (\rho_{j+1}^{(n)} + \rho_{j-1}^{(n)}) + \frac{1}{2} c (\rho_{j+1}^{(n)} - \rho_{j-1}^{(n)})$$

$$\Rightarrow \boxed{A = \cos \theta - iC \sin \theta}$$

Stable when  $|c| \leq 1$

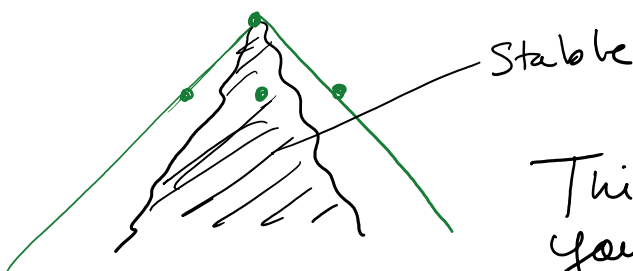
$$\text{or } \frac{|u| \Delta t}{\Delta x} < 1$$

Courant Condition or CFL Condition.

$$|u| < \frac{\Delta x}{\Delta t}$$

$\nearrow$  Physical Velocity       $\nwarrow$  Grid Velocity

Physical information must not travel faster than the maximum velocity on the grid.

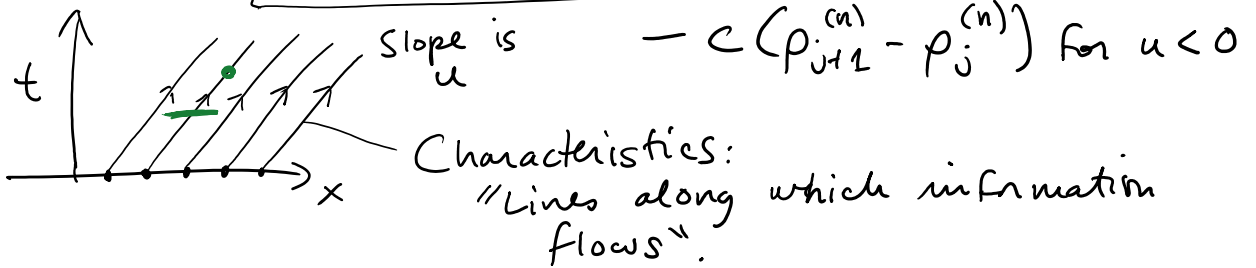


This is also the best you can do; you always have a CFL condition in

# hyperbolic Systems.

Upwind Scheme:

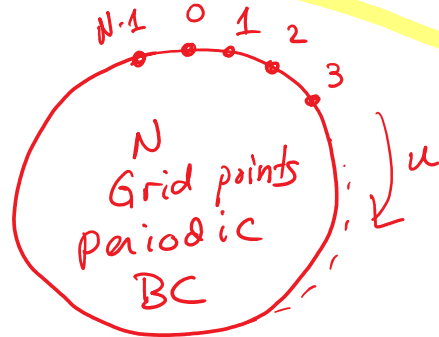
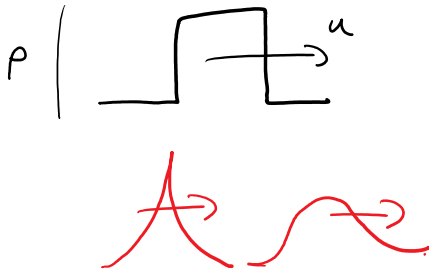
$$\rho_j^{(n+1)} = \rho_j^{(n)} - c(\rho_j^{(n)} - \rho_{j-1}^{(n)}) \quad \text{for } u > 0$$



Lax - WENDroff Method:

$$\rho_j^{(n+1)} = \frac{1}{2}c(1+c)\rho_{j-1}^{(n)} + (1-c^2)\rho_j^{(n)} - \frac{1}{2}c(1-c)\rho_{j+1}^{(n)}$$

for  $u > 0$



Fixed  
(sorry)