Hydrodynamics in 1-D

07 December 2020 12:58

Oral Exam will take place on the same Zoom as lecture. ~20 min

Recall: LAX Method:

correction!

$$\rho_i^{(n+1)} = \frac{1}{2}\left(\rho_{i+1}^{(n)} + \rho_{i-1}^{(n)}\right) - \frac{1}{2} C \left(\rho_{i+1}^{(n)} - \rho_{i-1}^{(n)}\right)$$

$$C = \frac{\Delta t\, u}{\Delta x}$$

For linear advection equation.

Here the flux is $f(\rho) = \rho u$

Rewrite this LAX Method using fluxes gives:

$$\rho_i^{(n+1)} = \frac{1}{2}\left(\rho_{i+1}^{(n)} + \rho_{i-1}^{(n)}\right) - \frac{\Delta t}{2\Delta x}\left(f_{i+1}^{(n)} - f_{i-1}^{(n)}\right)$$

Rewrite Upwind Method:

$$\rho_i^{(n+1)} = \rho_i^{(n)} - \frac{\Delta t}{\Delta x}\left(f_i^{(n)} - f_{i-1}^{(n)}\right) \quad \text{for } u \geq 0$$

$$= \rho_i^{(n)} - \frac{\Delta t}{\Delta x}\left(f_{i+1}^{(n)} - F_i^{(n)}\right) \quad \text{for } u < 0$$

Now for 1-D hydrodynamics we have 3 conservation equations;

1. Conservation of mass : $F = \rho u$
2. Conservation of momentum : $F = \rho u^2 + P$
3. Conservation of Energy : $F = u(E+P)$

Where P is the pressure

Now let the "State Vector" be $\underline{U} = \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}$

$$\underline{E}(\underline{U}) = \left[\rho u\ ,\ \rho u^2 + P\ ,\ u(E+P)\right]$$

little u is the velocity in x-direction

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$$\boxed{\frac{\partial \underline{U}}{\partial t} + \nabla \cdot \underline{F}(\underline{U}) = 0}$$

Very general conservation of "stuff"

We have 3 equations, but 4 unknowns: ρ, u, P, E!

E = Kinetic Energy + Thermal Energy Density

$$E = \frac{1}{2}\rho u^2 + \rho e$$

e: thermal energy per unit mass (specific thermal energy)

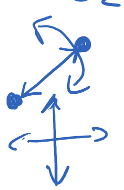
We still have 4 unknowns: ρ, u, P, e

- But we can introduce an equation of state (EOS) for the material.
- We will choose an ideal gas EOS:

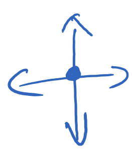
$$e = \frac{P}{(\gamma - 1)\rho} \quad \text{Ideal gas}$$

$$\gamma = \frac{f+2}{f}$$ where f is the number of degrees of freedom of the particles in the gas

O_2



$f = 4$



2-D Monoatomic gas

$$f = 2 \Rightarrow \gamma = 2$$

in 1-D: $\gamma = 3$

1-D Mono now

$$E = \frac{1}{2}\rho u^2 + \frac{1}{2}P$$

$$F(U) = \left[\rho u, \rho u^2 + P, u\underbrace{\left(\frac{1}{2}\rho u^2 + \frac{3}{2}P\right)}_{E+P}\right]$$

In finite volume let the value of $\underline{U}$ in each cell be the average of the "material" in the cell:

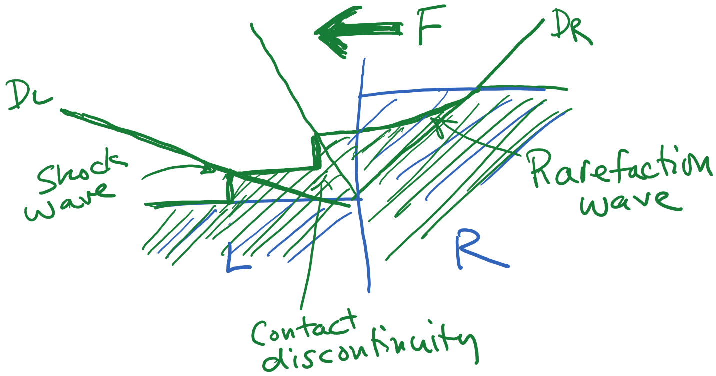
$$\langle U \rangle_i^n = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} U(x, t^n)\, dx$$

More complicated than for lin. advection because information flows in both directions.

$$\frac{\partial \underline{U}}{\partial t} + \nabla \cdot \underline{F}(\underline{U}) = 0$$

$$\frac{\langle U \rangle_i^{n+1} - \langle U \rangle_i^n}{\Delta t} \quad \frac{F_{i+\frac{1}{2}}^{n+1/2} - F_{i-\frac{1}{2}}^{n+1/2}}{\Delta x} = 0$$

Given that we have one state on the left and one state on the right, how do we approximate the flux through the boundary between them?



We average everything under the green curve again.

3 Waves (Characteristics) for 3 conservation laws!

$$F(\underline{U}_L, \underline{U}_R) \leftarrow \text{Riemann Solver}$$

$D_L, D_R \leftarrow$ Discontinuity speed to the left and to the right.

Lax-Friedrichs Riemann Solver:

Makes the simplification by assuming that $D_L = -D_R$; set to just the maximum discontinuity speed D_{max}

$$F^n_{i-\frac{1}{2}} = \frac{1}{2}\left(F^n_i + F^n_{i-1}\right) - \frac{1}{2} D_{max}\left(\underline{U}^n_{\boxed{i}} - \underline{U}^n_{\boxed{i-1}}\right)$$

Right Left

$$D = |u| + C_s$$

$$C_s = \sqrt{\gamma \frac{P}{\rho}} \quad \text{speed of sound.}$$

$$D_{max} = \max\left(D_{L_{i-1}}, D_{R_i}\right)$$

$$\boxed{\underline{F}^n_{i-\frac{1}{2}} = \frac{1}{2}\left[\underline{F}(\underline{U}^n_i) + \underline{F}(\underline{U}^n_{i-1})\right] - \frac{1}{2}\max\left(|u_{i-1}| + C_{s_{i-1}}, |u_i| + C_{s_i}\right)\left[\underline{U}^n_i - \underline{U}^n_{i-1}\right]}$$

Back to our time-space centered approximation of the averages in cells:

$$\underline{U}^{n+1}_i = \underline{U}^n_i - \frac{\Delta t}{\Delta x}\left[F^{n+1/2}_{i+\frac{1}{2}} - F^{n+1/2}_{i-\frac{1}{2}}\right]$$

simpler is to use n here

Method B

How do we get F at n+1/2

We make a predictor step to n+1/2 in order to calculate these fluxes:

$$^*\underline{U}^{n+1/2}_i = \underline{U}^n_i - \frac{\frac{1}{2}\Delta t}{\Delta x}\left[F^n_{i+1/2} - F^n_{i-1/2}\right] \quad \text{predictor step}$$

follow this with the update step.

$$\underline{U}^{n+1}_i = \underline{U}^n_i - \frac{\Delta t}{\Delta x}\left[F\left(^*U^{n+1/2}_i, {}^*U^{n+1/2}_{i+1}\right) - \ldots\right]$$

$$F\left({}^{*}U_{i-1}^{n+1/2}, {}^{*}U_{i}^{n+1/2}\right)\Big]$$

Method $\boxed{C}$

Method $\boxed{A}$: the original LAX Method

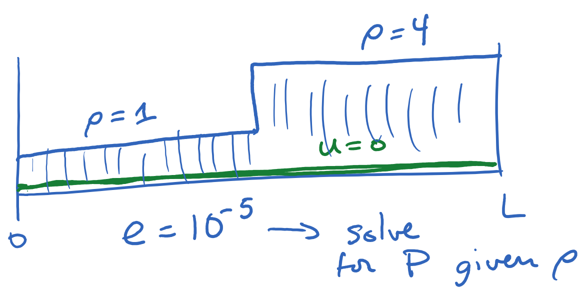
$$\underline{U}_i^{n+1} = \frac{1}{2}\left(\underline{U}_{i+1}^{n} + U_{i-1}^{n}\right) - \frac{\Delta t}{2\Delta x}\left(F_{i+1}^{n} - F_{i-1}^{n}\right)$$

should work, but won't conserve anything.

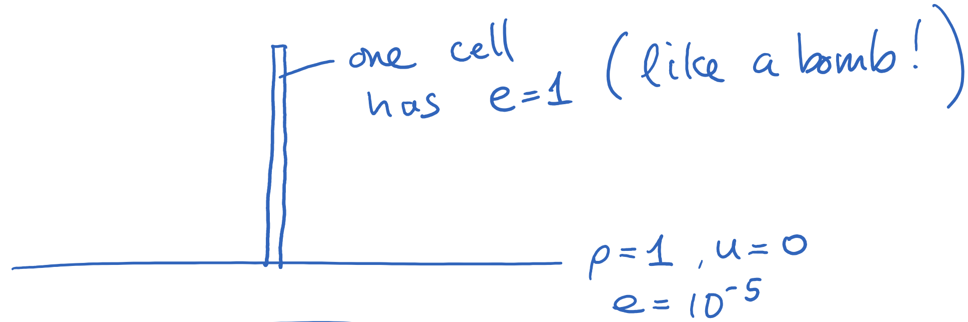
Exercise : Make a 1-D code with periodic B.C.s like previously but now with the state vector $\underline{U}$ in each cell.

2 ICs for each of Methods $\boxed{A}$, $\boxed{B}$ $\boxed{C}$

Try a "shock tube" problem :



Try a Sedov-Taylor Blast wave!



$\Delta t = ?$ Can use $D_{max} < \frac{\Delta x}{\Delta t}$

Courant Condition