ESC201 : Simulations in the Natural Sciences

www.ics.uzh.ch/~stadel
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~14:00 Exercise takes place in Teams

Formulas are great, but algorithms are better!

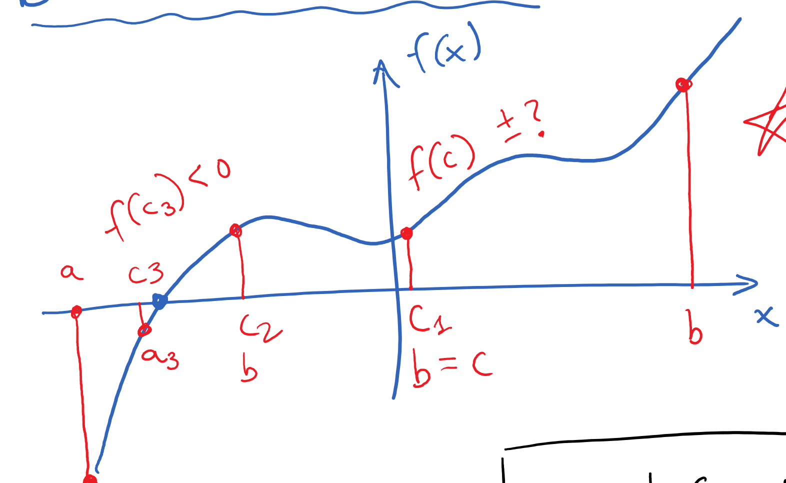
$x^x - 100 = 0$

$ax^3 + bx^2 + \cdots + d = 0$ ✓

$ax^4 + bx^3 + cx^2 + \cdots + e = 0$ ✓

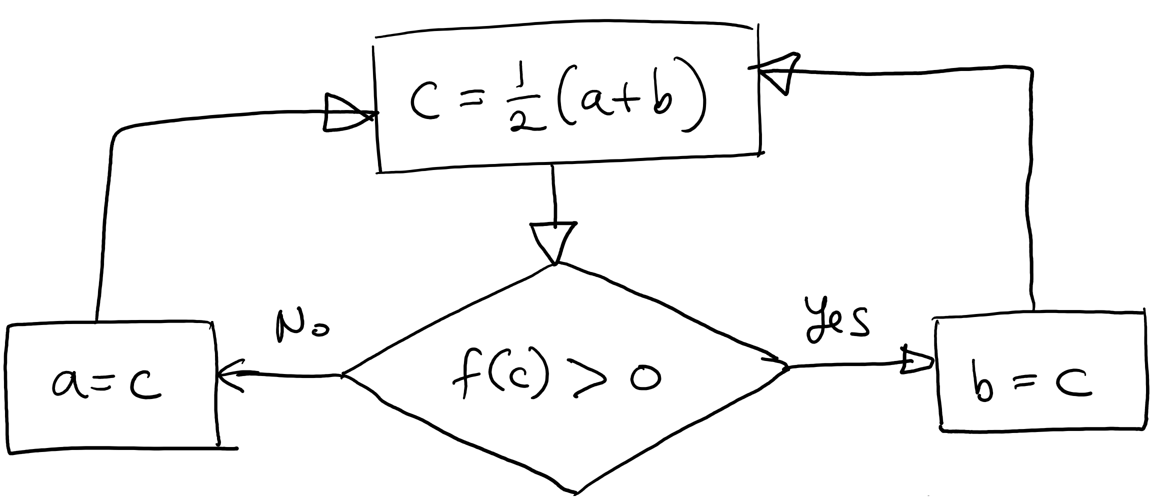
$ax^5 + \cdots + f = 0$

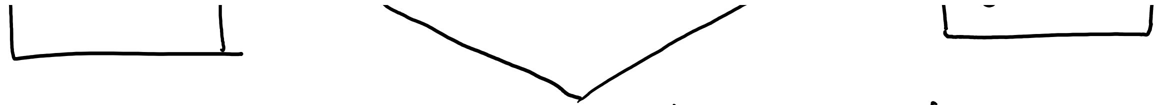
Bisection Method



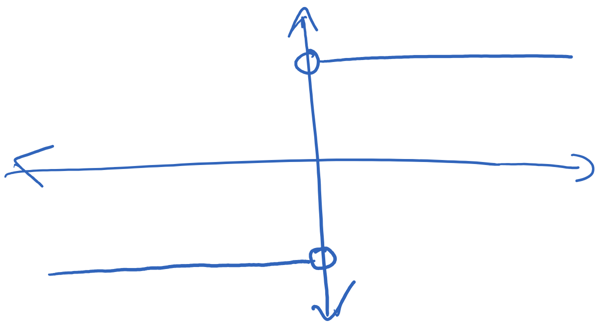
$f(x) = 0$

☆ Bracketing the Root!





When the initial $f(a)$ is positive and $f(b)$ is negative then we swap a and b at the beginning!



$$|a-b| < \varepsilon_{absolute}$$

$$\frac{|a-b|}{|c|} < \varepsilon_{relative}$$

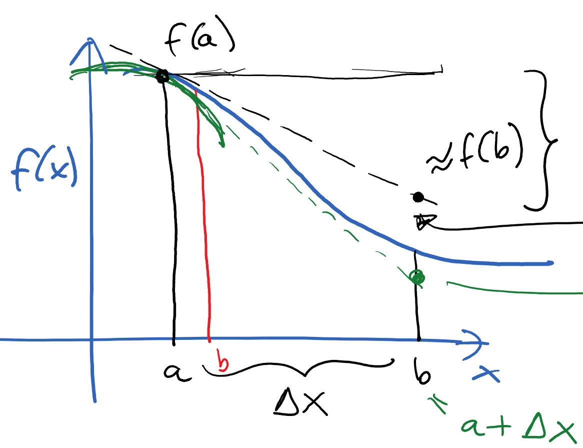
Half the interval with every iteration. So you gain 1 binary digit per iteration.

• could also be k digits/step

$\longrightarrow$ Linear convergence

Taylor expansion

$$f'(a) = \frac{\partial f}{\partial x}\Big|_a$$



$\Delta x \cdot f'(a)$

$f(a) + \Delta x f'(a) \approx f(b)$

$f(a) + \Delta x f'(a) + \frac{\Delta x^2}{2} \cdot f''(a)$

$$f(b) \equiv f(a+\Delta x) \cong f(a) + \Delta x f'(a) + \frac{1}{2}\Delta x^2 f''(a)$$

$$+\frac{1}{6}\Delta x^3 f^{(3)}(a) + \cdots + \frac{1}{n!}\Delta x^n f^{(n)}(a)$$

$$+\left[\frac{\Delta x^n}{(n-1)!}\int_0^1 (1-t)^{n-1} f^{(n)}(a+t\Delta x)\,dt\right]$$

Sometimes it is possible to find a practical and relatively tight upper bound to this integral.

Let's ignore all but the 1st order term!

$$f(x) = f(a+\Delta x) \cong f(a) + \Delta x f'(a)$$

Let's suppose this function has a root at x : $f(x) = 0$

$$\boxed{a + \Delta x \equiv x}$$

$$0 \cong f(a) + \Delta x \cdot f'(a)$$

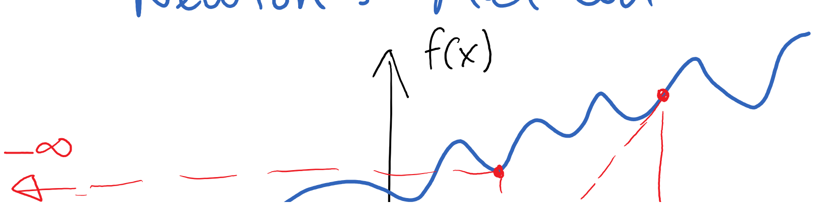
$$\Delta x \cong -\frac{f(a)}{f'(a)}$$

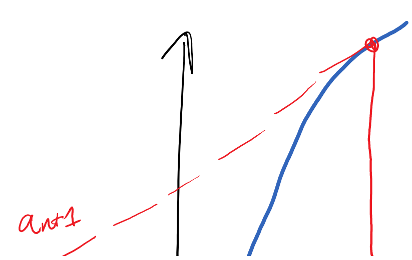
$$\boxed{x \cong a - \frac{f(a)}{\boxed{f'(a)}}}$$

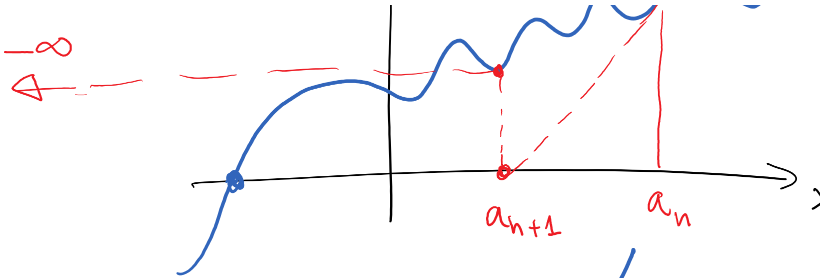
!

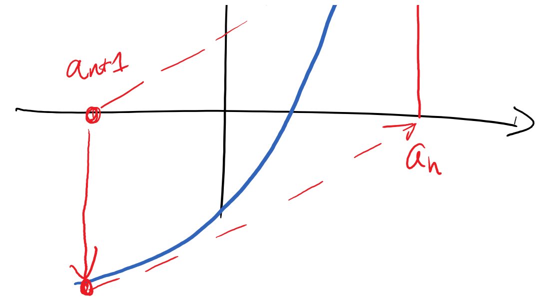
$$\boxed{a = x}$$

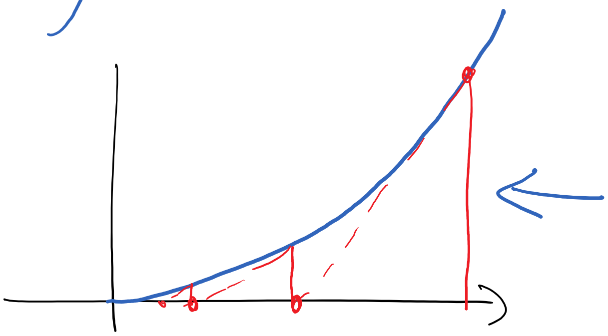
Newton's Method











Sometimes you can gaurantee convergence and stability.

Today's Assignment: Kepler's Equation

$$M = E - e \sin E$$

also angle — an angle

Need to find E given M and e.

$f(E) = 0$ $\qquad \boxed{f(E) = E - e\sin E - M}$

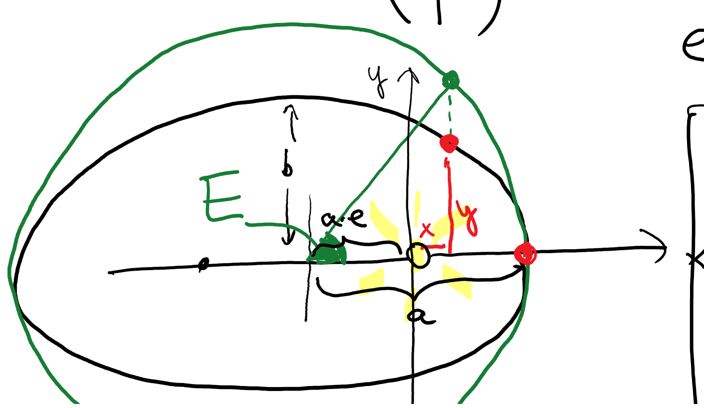
$$\boxed{M = nt} \qquad n = \frac{2\pi}{T} \text{ – period Years} \qquad T = a^{3/2}$$

years — A.U.

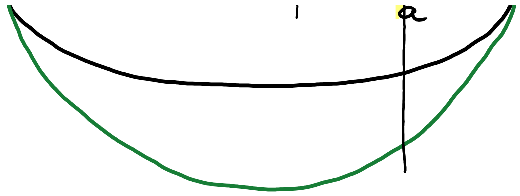
or

$$M = 2\pi\left(\frac{t}{T}\right)$$

$e = 0 \Rightarrow$ circle



$$\boxed{\begin{aligned} x &= a\cdot\cos(E) - ae \\ y &= b\sin(E) \\ &= a\sqrt{1-e^2}\,\sin(E) \end{aligned}}$$



$$= a\sqrt{1-e^2}\,\sin(E)$$

$e = 0.5 \qquad a = 1$

$f'(E) = ?$