MacCormack's method

$\Rightarrow$ well-behaved numerical solutions to non-linear hyperbolic PDEs

recap: ESC 201

lower order

2nd order PDEs

$$\boxed{A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial y \partial x} + C \frac{\partial^2 u}{\partial y^2} \left(+ E \frac{\partial u}{\partial x} \dots \right) = 0}$$

if $B^2 - AC < 0$: elliptic PDE, eg. Poisson Equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial x^2} = f(x,y)$$

- Jacobi, SOR, FFT, Multigrid

if $B^2 - AC = 0$: parabolic PDE, eg. Diffusion Equation

$$\frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2}$$

- Crank-Nicholson

if $B^2 - AC > 0$: hyperbolic PDE, e.g. Wave Equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

, e.g. 1D-advection

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0$$

- LAX, Upwind, LAX-Wendroff
  (these are finite difference methods)
- Godunov
  (finite volume method)

Let us recall specifically the Lax-Wendroff method:

For the 1D-advection Equation:

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0 \qquad (1)$$

To solve for $u(x,t)$, discretize on a grid u_i^n and Taylor expand:

$$u_i^{n+1} = u_i^n + \Delta t \left.\frac{\partial u}{\partial t}\right|_i^n + \frac{(\Delta t)^2}{2} \left.\frac{\partial^2 u}{\partial t^2}\right|_i^n + \ldots$$

From (1) $\Rightarrow$ $\left.\frac{\partial u}{\partial t}\right|_i^n = -a \left.\frac{\partial u}{\partial x}\right|_i^n \quad (2)$

$$\left.\frac{\partial^2 u}{\partial t^2}\right|_i^n = a^2 \left.\frac{\partial^2 u}{\partial x^2}\right|_i^n \quad (3)$$

Approximate derivatives with central differences:

$$\left.\frac{\partial u}{\partial x}\right|_i^n \simeq \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}, \qquad \left.\frac{\partial^2 u}{\partial x^2}\right|_i^n \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}$$

Inserting into (1):

$$u_i^{n+1} = u_i^n - \frac{c}{2}\left(u_{i+1}^n - u_{i-1}^n\right) + \frac{c^2}{2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

with $c = \frac{\Delta t a}{\Delta x}$

$$\boxed{u_i^{n+1} = \frac{c}{2}(1+c)\, u_{i-1}^n + (1-c^2)\, u_i^n - \frac{c}{2}(1-c)\, u_{i+1}^n}$$

This is accurate to 2nd order in Δx and Δt:

$T = \mathcal{O}(\Delta x^2, \Delta t^2)$

Stable if $|c| \leq 1$.

What if the problem is not linear?

In general:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \quad (4)$$

(for 1D-advection: $f(u) = a \cdot u$)

(for Euler equations, u and $f(u)$ are vectors, and the equation describes the conservation laws of mass, momentum and energy)

Solutions to this equation can have discontinuities appearing.

From (4):

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\partial^2 f(u)}{\partial t \partial x} = -\frac{\partial}{\partial x}\left(\frac{\partial f(u)}{\partial t}\right)$$

$$\frac{\partial f(u)}{\partial t} = \underbrace{\frac{\partial f(u)}{\partial u}}_{=A}\frac{\partial u}{\partial t} \quad , \quad \frac{\partial u}{\partial t} = -\frac{\partial f(u)}{\partial x} = -\overbrace{\frac{\partial f(u)}{\partial u}}^{=A}\frac{\partial u}{\partial x}$$

$$\Rightarrow \frac{\partial f(u)}{\partial t} = -A\frac{\partial f(u)}{\partial x}$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(A\frac{\partial f(u)}{\partial x}\right)$$

$A := \frac{\partial f(u)}{\partial u}$ the 'Jacobian'

If u and f are vectors, then A is a matrix $A_{ij} = \frac{\partial f_i(u)}{\partial u_j}$

Inserting these into the Taylor expansion from before:

$$u_i^{n+1} = u_i^n - \Delta t \left.\frac{\partial f(u)}{\partial x}\right|_i^n + \frac{(\Delta t)^2}{2}\frac{\partial}{\partial x}\left(A\left.\frac{\partial f(u)}{\partial x}\right|_i^n\right)\Bigg|_i^n + \dots$$

Central difference leads to:

$$\boxed{u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x}\frac{f_{i+1}^n - f_{i-1}^n}{2} + \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2\left[A_{i+1/2}^n\left(f_{i+1}^n - f_i^n\right) - A_{i-1/2}^n\left(f_i^n - f_{i-1}^n\right)\right]}$$

with $A_{i+1/2} = A\left(\frac{u_i + u_{i+1}}{2}\right)$

The Lax-Wendroff method requires calculating a lot of entries for the Jacobian.

The idea behind MacCormack's method is to avoid carrying the 2nd order derivatives (which introduce the Jacobian terms), while retaining 2nd order accuracy in Δt and Δx.

$$u_i^{n+1} = u_i^n + \left(\frac{\partial u}{\partial t}\right)_{av} \Delta t$$

where $\left(\frac{\partial u}{\partial t}\right)_{av}$ is a representative mean time derivative between the times t and $t+\Delta t$:

$$\left(\frac{\partial u}{\partial t}\right)_{av} = \frac{1}{2}\left[\left(\frac{\partial u}{\partial t}\right)_i^n + \left(\frac{\partial u}{\partial t}\right)_i^{\overline{n+1}}\right]$$

a predicted value for $\frac{\partial u}{\partial t}$ at $t+\Delta t$

$\Rightarrow$ The MacCormack method is a Predictor-Corrector method:

① Predictor step: $\left(\frac{\partial u}{\partial t}\right)_i^n = -\frac{f_{i+1}^n - f_i^n}{\Delta x}$

(forward difference in space)

$$\boxed{u_i^{\overline{n+1}} = u_i^n - \frac{\Delta t}{\Delta x}\left(f_{i+1}^n - f_i^n\right)}$$

② Corrector step: $\left(\frac{\partial u}{\partial t}\right)_i^{\overline{n+1}} = \cancel{f_{i+}} - \frac{f_i^{\overline{n+1}} - f_{i-1}^{\overline{n+1}}}{\Delta x}$

(backward diff. in space)

$$\boxed{u_i^{n+1} = u_i^n - \frac{\Delta t}{2\Delta x}\left[\left(f_{i+1}^n - f_i^n\right) + \left(f_i^{\overline{n+1}} - f_{i-1}^{\overline{n+1}}\right)\right]}$$

The forward and backward differences can be interchanged, it works best when the predictor difference is in same direction as the propagation direction

Apply MacCormack's method to 1D-advection:

$$f(u) = au$$

$$u_i^{n+1} = u_i^n - \frac{c}{2}\left[u_{i+1}^n - u_i^n + \overline{u_i}^{n+1} - \overline{u_{i-1}}^{n+1}\right]$$

$$= u_i^n - \frac{c}{2}\left[u_{i+1}^n - u_i^n + u_i^n - c\left(u_{i+1}^n - u_i^n\right) - u_{i-1}^n + c\left(u_i^n - u_{i-1}^n\right)\right]$$

$$\boxed{u_i^{n+1} = \frac{c}{2}(1+c)\,u_{i-1}^n + (1-c^2)\,u_i^n - \frac{c}{2}(1-c)\,u_{i+1}^n}$$

This is exactly identical to Lax-Wendroff scheme (for linear problem) but avoids Jacobian calculations.

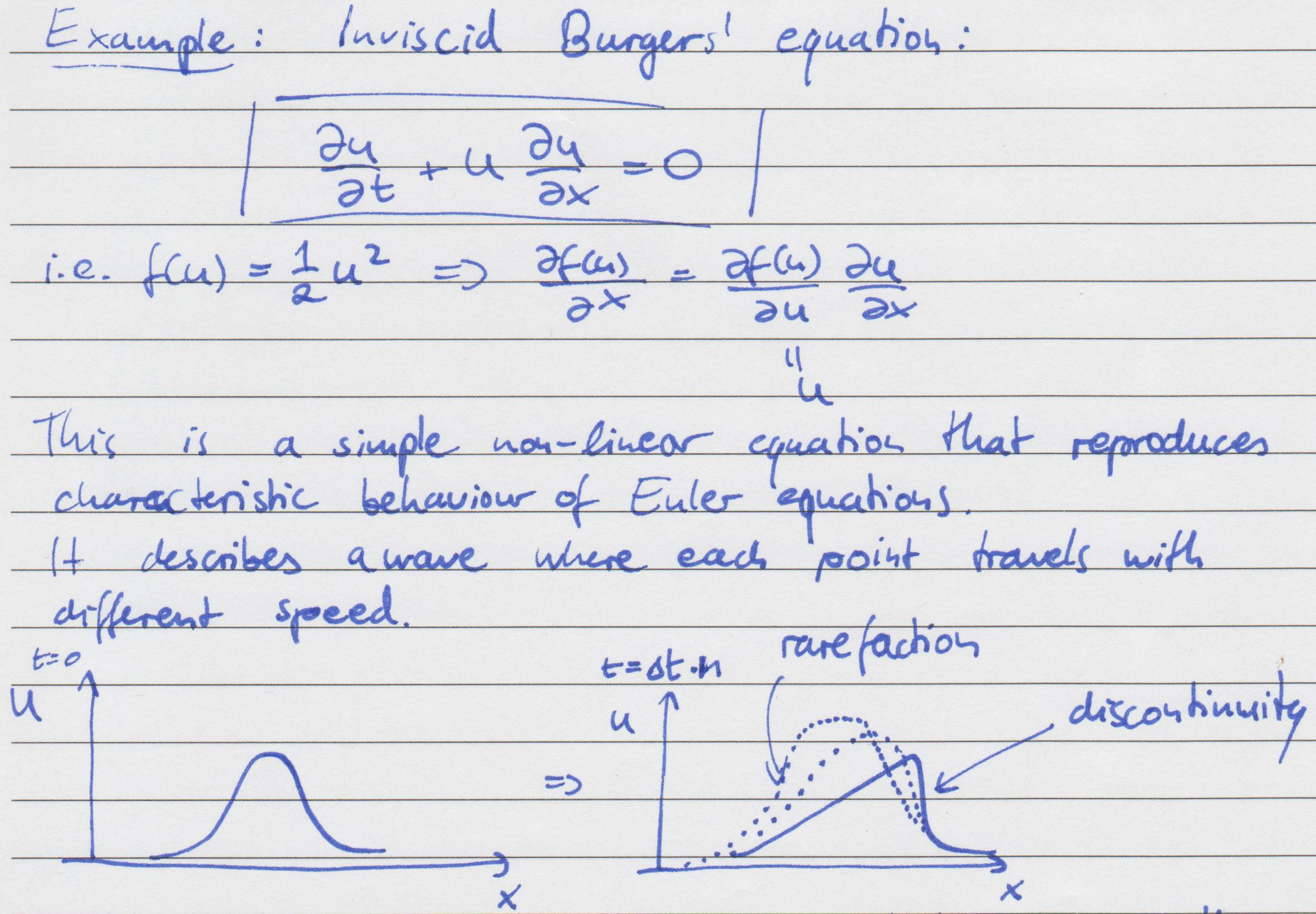
What about non-linear problems?

<u>Example</u>: Inviscid Burgers' equation:

$$\boxed{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0}$$

i.e. $f(u) = \frac{1}{2}u^2 \Rightarrow \frac{\partial f(u)}{\partial x} = \underbrace{\frac{\partial f(u)}{\partial u}}_{=u}\frac{\partial u}{\partial x}$

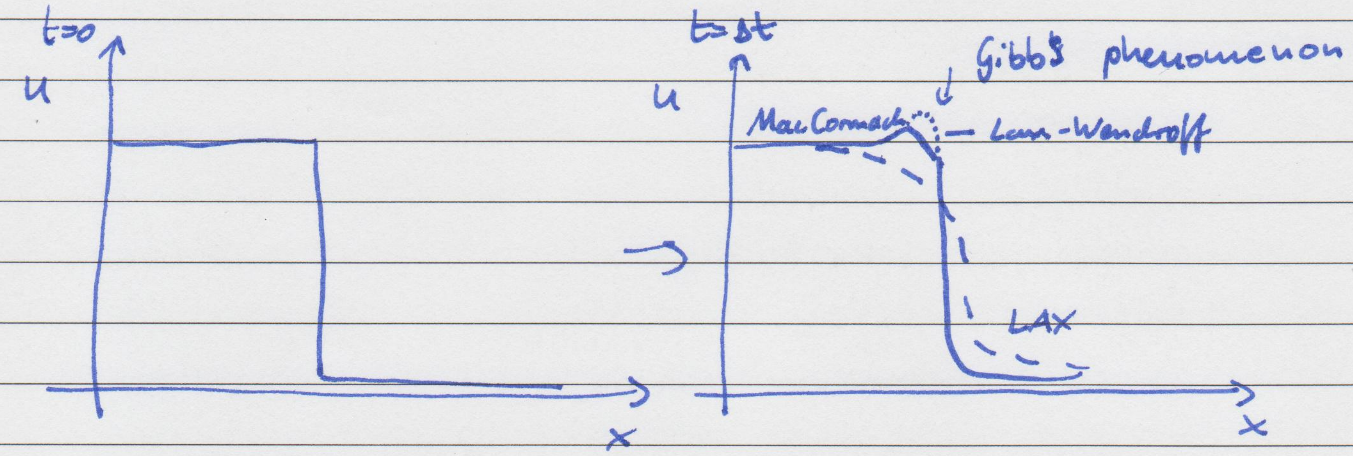
This is a simple non-linear equation that reproduces characteristic behaviour of Euler equations.
It describes a wave where each point travels with different speed.



MacCormack's method should be slightly better than Lax-Wendroff here. Best at $C=1$.

Von Neumann-analysis:
MacCormack's method is stable if $\frac{u_{max} \Delta t}{\Delta x} \leq 1$.

It introduces dispersive errors into the solution, but no diffusive errors (check modified equations!)



Maybe next week: Use MacCormack's method to simulate sub-sonic super-sonic isentropic nozzle flow: