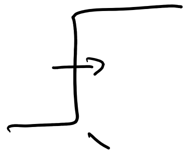
Finite Difference

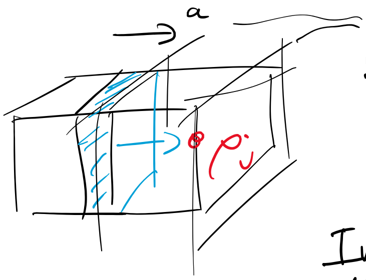
$$\frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} = 0$$

operators on a grid



$\frac{\partial \rho}{\partial x} \to \infty$

Conservation $\leadsto$ Integral Equation



how mass flows through those boundaries?

$$\rho_j^{n+1} = \rho_j^n + \frac{\Delta t}{\Delta x}\left[f_{j-\frac{1}{2}} - f_{j+\frac{1}{2}}\right]$$

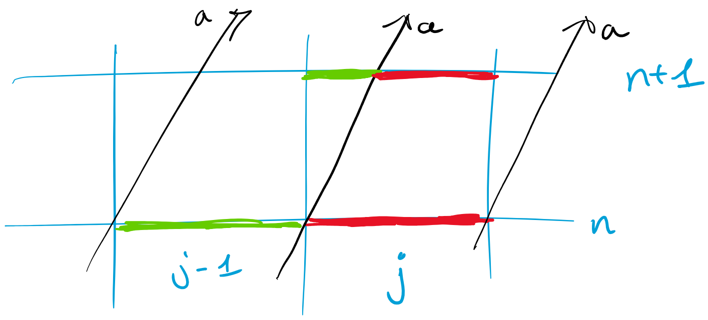
wall's coordinate

Integration over all of the fluxes should be exactly $\emptyset$.

Exact conservation

Simple Case: Linear Advection

$$f(\rho) = a\cdot\rho \qquad a \geq 0$$



$$\rho_j^{n+1} = \underbrace{\left(\frac{a\cdot\Delta t}{\Delta x}\right)}_{C}\rho_{j-1}^n + \left(1 - \frac{a\cdot\Delta t}{\Delta x}\right)\rho_j^n$$

$$\rho_j^{n+1} = C\cdot\rho_{j-1}^n + (1-C)\rho_j^n \qquad a \geq 0$$

linear interpolation

$$\rho_j^{n+1} = \rho_j^n - c\left(\rho_j^n - \rho_{j-1}^n\right)$$

Finite Volume Method: Godunov's Method

1^{st}-order upwind method.

Modified Equation: It is the continuous equation that our numerical/discretized

Modified equation: ... the continuous equation that our numerical/discretized method is actually following.

USELESS Method

$$\frac{\rho_j^{n+1}-\rho_j^{n}}{\Delta t} + a\,\frac{\rho_{j+1}^{n}-\rho_{j-1}^{n}}{2\Delta x} = 0$$

Recall: this is unstable

Taylor expand about "t" or step "n"

$\rho(x_j, t+\Delta t)$

$$\rho_j^{n+1} = \cancel{\rho_j^{n}} + \Delta t\left(\frac{\partial \rho}{\partial t}\right) + \frac{\Delta t^2}{2}\left(\frac{\partial^2 \rho}{\partial t^2}\right) + \dots$$

$\rho(x_j, t)$

$$\rho_{j+1}^{n} \equiv \rho(x_j+\Delta x, t) \rightarrow \text{expand} \dots$$

$$\rho_{j+1}^{n} = \cancel{\rho_j^{n}} + \Delta x\left(\frac{\partial \rho}{\partial x}\right) + \cancel{\frac{\Delta x^2}{2}\left(\frac{\partial^2 \rho}{\partial x^2}\right)} + \dots$$

$$\rho_{j-1}^{n} = \cancel{\rho_j^{n}} - \Delta x\left(\frac{\partial \rho}{\partial x}\right) + \cancel{\frac{\Delta x^2}{2}\left(\frac{\partial^2 \rho}{\partial x^2}\right)} + \dots$$

$$\frac{\cancel{\Delta t}\left(\frac{\partial \rho}{\partial t}\right) + \frac{\Delta t^{\cancel{2}}}{2}\left(\frac{\partial^2 \rho}{\partial t^2}\right)}{\cancel{\Delta t}} + a\,\frac{\left(\cancel{2\Delta x}\left(\frac{\partial \rho}{\partial x}\right)\right)}{\cancel{2\Delta x}} = 0$$

$$\boxed{\frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} = -\frac{\Delta t}{2}\left(\frac{\partial^2 \rho}{\partial t^2}\right)}$$

$$\frac{\partial}{\partial t}: \quad \frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} = 0 \Rightarrow \frac{\partial^2 \rho}{\partial t^2} + a\frac{\partial}{\partial x}\left(\frac{\partial \rho}{\partial t}\right) = 0$$

$$\frac{\partial^2 \rho}{\partial t^2} = -a\frac{\partial}{\partial x}\left(\frac{\partial \rho}{\partial t}\right)$$

$$\frac{\partial \rho}{\partial t} = -a\frac{\partial \rho}{\partial x}$$

$$= +a^2\frac{\partial^2 \rho}{\partial x^2}$$

$$\frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} = -\frac{\Delta t\, a^2}{2}\left(\frac{\partial^2 \rho}{\partial x^2}\right)$$

$$\frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} = \boxed{-\frac{\Delta t\, a^2}{2}}\left(\frac{\partial^2 \rho}{\partial x^2}\right)$$

D it is negative! Unstable

Homework: What is the modified equation of C.I.R. Scheme (1st order Upwind)

2-D Advection

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\, \underline{u}) = 0$$

$$\underline{u} = \langle a, b\rangle \qquad a, b > 0$$

$$\frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} + b\frac{\partial \rho}{\partial y} = 0$$

1st order upwind

$$\frac{\rho_{jl}^{n+1} - \rho_{jl}^{n}}{\Delta t} + a\,\frac{\rho_{jl}^{n} - \rho_{j-1\,l}^{n}}{\Delta x} + b\,\frac{\rho_{jl}^{n} - \rho_{j\,l-1}^{n}}{\Delta y} = 0$$

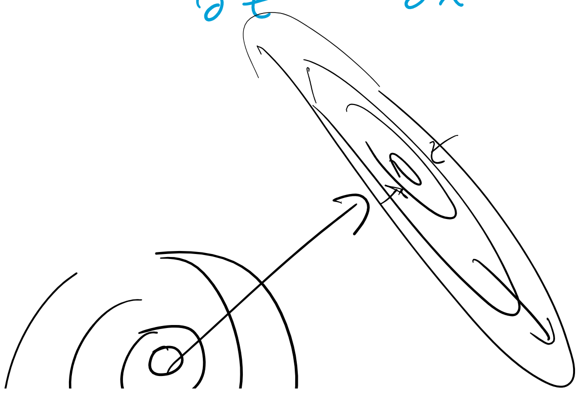
Stability Analysis shows that

$$C_a > 0 \qquad C_a = \frac{a\Delta t}{\Delta x} \qquad \boxed{C_a + C_b \leq 1}$$

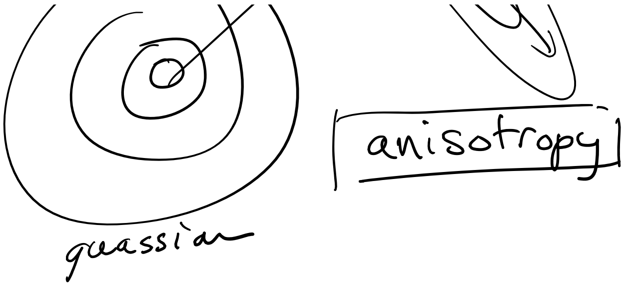
$$C_b > 0 \qquad C_b = \frac{b\Delta t}{\Delta y}$$

Modified Equation for this is interesting:

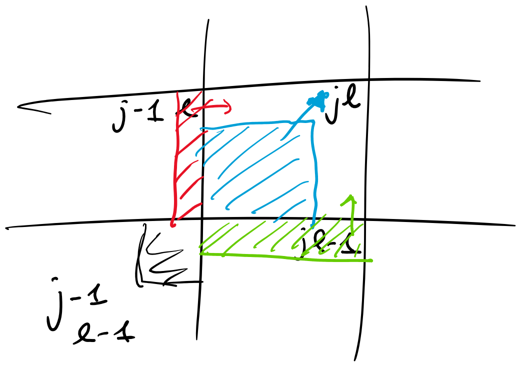
$$\frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} + b\frac{\partial \rho}{\partial y} = \frac{a\Delta x}{2}(1-C_a)\frac{\partial^2 \rho}{\partial x^2} + \frac{b\Delta y}{2}(1-C_b)\frac{\partial^2 \rho}{\partial y^2} \boxed{- ab\Delta t\frac{\partial^2 \rho}{\partial x \partial y}}$$



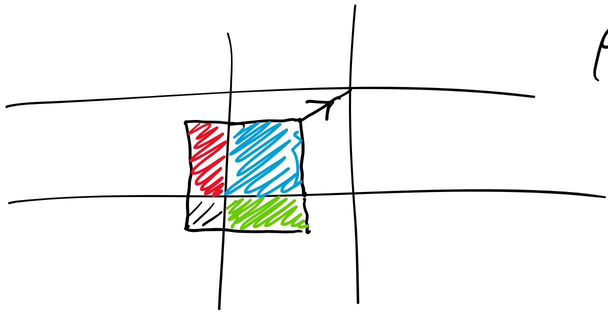
Cross derivative term won't preserve shape



Cross derivative term won't preserve shape



Corner Transport Upwind Method



$$\rho_{jl}^{n+1} = \boxed{(1-c_a)(1-c_b)}\,\rho_{jl}^{n} + \boxed{c_a(1-c_b)}\,\rho_{j-1\,l}^{n} + \boxed{c_b(1-c_a)}\,\rho_{j\,l-1}^{n} + \boxed{c_a c_b}\,\rho_{j-1\,l-1}^{n}$$

Stability is also improved

$$\boxed{\begin{array}{l} 0 \leq c_a < 1 \quad \text{and} \\ 0 \leq c_b < 1 \end{array} \;!}$$

Nice 2-step implementation:

$$\boxed{\begin{aligned} \rho_{jl}^{*} &= (1-c_a)\rho_{jl}^{n} + c_a\,\rho_{j-1\,l}^{n} \\ \rho_{jl}^{n+1} &= (1-c_b)\,\rho_{jl}^{*} + c_b\,\rho_{j\,l-1}^{*} \end{aligned}}$$

Start with a gaussian:

$$\rho(x,y) = A\exp\left[-\left(\frac{(x-x_0)^2}{2\sigma_x^2} + \frac{(y-y_0)^2}{2\sigma_y^2}\right)\right]$$

$$\rho(x,y) = A \exp\left[-\left(\frac{\quad}{2\sigma_x^2} + \frac{\quad}{2\sigma_y^2}\right)\right]$$

$$\sigma_x = \sigma_y$$

$$\boxed{\text{C.I.R.} \quad \text{vs} \quad \text{C.T.U}}$$